MATHEMATICAL MODELLING AND IDENTIFICATION OF A BATCH BALL MILL GRINDING SYSTEM

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By
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to the

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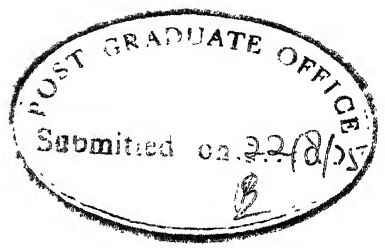
to

my father



CERTIFICATE

Certified that this work on 'Mathematical Modelling and Identification of a Batch Ball Mill Grinding System' , has been carried out under my supervision and that it has not been submitted elsewhere for a degree .

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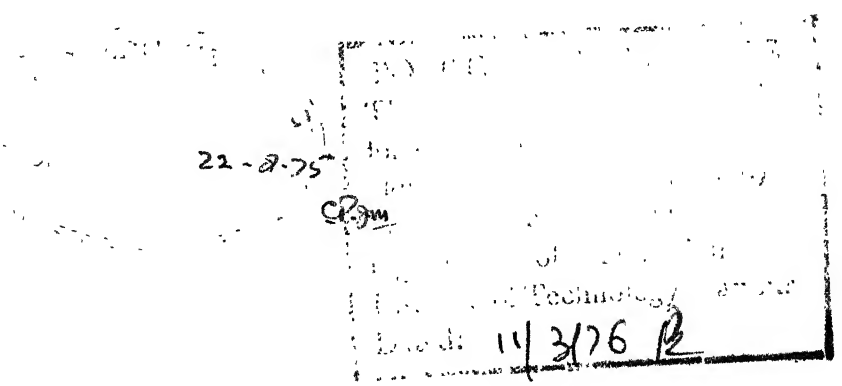

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TABLE OF CONTENTS

LIST OF TABLES

Page

LIST OF FIGURES

Page

Page

Page

LIST OF SYMBOLS

a	an exponent to time t in Eq. (2.63)
$\bar{a}$	an exponent to the first moment of particle size distribution in Eq. (2.68)
$a_0, a_1, \ldots a_5$	coefficients of the polynomial in Eq. (8.7)
$\bar{a}_1, \bar{a}_2$	system constants in Eq. (2.69)
$\bar{a}_3, \bar{a}_4$	system constants in Eq. (2.70)
A	a system constant associated with the expression for the selection function
$b_0, b_1, \ldots b_5$	coefficients of the polynomial in Eq. (8.8)
$b_{i,i}$	inner breakage parameter
$b_{i,j}$	distribution parameter
$b_{i,i}(t)$	time dependent inner breakage parameter
$b_{i,j}(t)$	time dependent distribution parameter
B	ball diameter
$B(v,x)$	breakage density function
$\bar{B}(v,x)$	breakage distribution function
$\bar{B}(x/v)$	breakage function which depends on only the size ratio x/v
$\bar{B}(v,x,\delta)$	Reid's modified breakage function, which is function of the sieve size ratio δ,
B^o	a constant in Eq. (10.11)

B_i^*	ball diameter for the maximum value of the rate parameter S_i
$\underset{=}{B}$	a lower triangular breakage parameter matrix
$B_{i,j}$	breakage parameter
$B_{i,j}(t)$	time dependent breakage parameter
$B_{i,j}(k)$	breakage parameter for k-th time interval
$\bar{B}_{i,j}$	cumulative breakage parameter
$\bar{B}_{i-j}$	difference - similar breakage parameter
$c_1, c_2, .. c_5$	coefficients of the polynomial in Eq. (8.11)
$c_{i,j}$	an element of $\underset{=}{C}$ matrix
$C(v,x)$	product particle size distribution corresponding to one breakage cycle
$\underset{=}{C}$	breakage matrix corresponding to one breakage cycle
C_0	defined in Eq. (2.60)
C_1	a constant in Eq. (2.1)
C_2	a constant in Eq. (2.5)
C_3	a constant in Eq. (2.7)
C_4	a constant in Eq. (2.8)
C_5	a constant in Eq. (2.11)
C_6	a constant in Eq. (2.62a)

d	a constant
$d_1, d_2, \ldots d_5$	coefficients of polynomial in Eq. (8.12)
$d_i(t)$	defined in Eq. (6.2)
$\underline{\underline{D}}$	input-output matrix for one breakage cycle, defined in Eq. (2.26)
$D_q(x/v)$	a composite distribution function defined in Eq. (2.22) for q-th primary breakage event
e	an exponent to the matrix $\underline{\underline{C}}$ in Eq. (2.31)
er	an error function defined in Eq. (2.41)
E	total specific energy of breakage applied to the particles during the course of grinding (energy responsible for fracture inducing environment)
ΔE	change in E
E'	net energy input to the charge
$\bar{E}$	the energy required to produce half the maximum surface area
E_g	gross energy input
E_r	an error function defined in Eq. (8.10)
$\underline{f}$	feed size distribution vector
$f(x,t)$	defined in Eq. (2.44)
$f(E')$	defined in Eq. (2.9)

$_1F_1$	the confluent hypergeometric function
$F(x)$	fraction finer than size x
$F(Q)$	retardation factor defined in Eq. (2.13)
$F(x,t)$	fraction finer than size x at time t
$F_i(t)$	cumulative fraction of total particulate mass finer than i-th size interval at time t
$F_q(x)$	fraction of material finer than size x after q primary breakage events.
$g(x)$	a monotonically non-decreasing function in particle size x
$g_1(x)$	defined in Eq. (4.14)
$g_2(x)$	defined in Eq. (4.16)
$\bar{g}(x)$	an unspecified function in particle size x
$G(v,x)$	fractional rate of formation of particles of size x from the primary breakage of particles of size v.
$G'(v,x)$	defined in Eq. (2.47)
G_i	defined in Eq. (2.113)
h	defined in Eq. (2.61)
$h(a',b';c';d',e')$	the generalized hypergeometric series in two variables
$h(v)$	a positive definite function over the range $0 < x < x_0$
$h_{i,j}$	defined in Eqs. (2.96), (2.97) and (2.98)
H^0	value of function H for W equal to reference particulate mass

$H(W)$	a function describing the variation of the rate parameters with particulate mass W
$H'(W)$	ratio of the functional values of H for a given particulate mass and the reference particulate mass
$H_i(a'',b'';c'',d'')$	defined in Eq. (2.46)
i	index for size interval
$\underline{\underline{I}}$	the identity matrix
j	index for size interval
k	index for time interval
k_o	a system constant in Eq. (2.119)
k_1, k_2	system constants in Eq. (10.9)
K	potential growth factor
K_1, K_2	system constants in Eq. (10.15)
K_3	a system constant in Eq. (10.16)
K_4, K_5	system constants in Eq. (10.17)
$L(v,x)$	selekage function defined in Eq. (5.18)
$L_1(v)$	one part of the separable selekage function
$L_2(x)$	reduced selekage function
$L_2(v,x,t)$	time dependent reduced selekage function defined in Eq. (7.20)
$\bar{m}$	a constant in Eq. (2.20)
m'	distribution modulus in Gate-Gaudin-Schumann size distribution equation
$M(x)$	particle mass density function in particle size x

$M(x,t)$	mass density of particulate solids at size x at time t
M_i	mass fraction of particulate solids in size interval i
$M_i(t)$	mass fraction of particulate solids in size interval i at time t
$M_i(t)$	experimental size distribution data
$\underline{M}(t)$	particle size distribution vector at time t
n	number of discrete size intervals
$\bar{n}$	a constant in Eq.(2.18)
$\underline{\underline{n}}$	a constant in Eq. (2.123) for the order of disappearance kinetics
$n_{i,j}$	an element of $\underline{\underline{N}}$ matrix
$\underline{\underline{N}}$	a matrix defined in Eq. (2.104)
$o_{i,j}$	elements of the input-output matrix $\underline{\underline{O}}$ corresponding to a uniformly distributed feed and a fixed period of grinding $\underline{\underline{t}}$
$o_{i,j}(k)$	elements of the input-output matrix $\underline{\underline{O}}(k)$ for k-th time interval
$\underline{\underline{O}}(k)$	input-output matrix for k-th time interval
$\underline{\underline{O}}(t)$	input-output matrix corresponding to a grinding period of t
p	a constant in Eq. (2.18)
$\bar{p}$	a dimensionless variable defined in Eq.(10.18)

$\underline{p}$	product size distribution vector
p_q	probability of breakage of particles during q-th breakage event
$p^*_{i,i}$	probability of breakage of particles in i-th size interval during one breakage cycle
$P(t)$	power draught to the mill at time t
$P(x_i,t)$	defined in Eq. (6.32)
$\underset{=}{P}^*$	a diagonal breakage probability matrix for one breakage cycle
$Q(t)$	total surface area of the particulate solids at time t
Q_∞	maximum surface area of the particulate solids corresponding to the grind limit
ΔQ	change in total surface area of the particulate solids
r	a constant introduced in Eq. (2.1)
$\bar{r}$	cumulative fraction of particulate mass coarser than size x_i after grinding time $t_i(\bar{r})$
r_v	Kick's law deviation exponent
$R(x,t)$	cumulative fraction of particulate solids coarser than size x at time t
$R_i(t)$	cumulative fraction of particulate solids coarser than size x_i at time t
s_i	selection parameter for -th size interval
$s_i(t)$	time dependent selection parameter for i-th

$S(x)$	selection function at size x
S_i	time invariant rate parameter for i-th size interval
$S_i(k)$	rate parameter for i-th size interval and k-th time interval
$S_i(t)$	rate parameter for i-th size interval at time t
$S_i(B)$	rate parameter for i-th size interval corresponding to ball diameter B
$S_i(W)$	rate parameter for i-th size interval corresponding to the total mass of particulate solids W
S_i^0	rate parameter for i-th size interval corresponding to reference particulate mass
S_i^*	maximum value of the rate parameter for i-th size interval
$\bar{S}_i$	a characteristic grinding rate constant for i-th size interval
$\underline{\underline{S}}$	a diagonal matrix of rate parameters S_i
$t, \bar{t}, t'$	grinding time
$\bar{\bar{t}}$	a fixed grinding time interval
$t_i(\bar{r})$	grinding time corresponding to $R_i = \bar{r}$
$t_i^0(\bar{r})$	grinding time corresponding to $R_i = \bar{r}$, under the condition that the particulate mass is equal to the reference particulate mass
t_{k-}	grinding time at the end of k-th time interval defined in Eq. (2.68)
	particle size

$\bar{u}$	pseudo particle size defined in Eq. (4.4)
$U_1(x_i)$	an unknown function describing the variation of B_i^* with x_i
$U_2(B_i^*, x_i)$	an unknown function describing the variation of S_i^* with x_i
v	size of the parent particle
$v_{i,i}$	specific velocity of breakage of the particles in the i-th size interval.
$v_{i,j}$	specific rate of appearance of particles in i-th size interval when particles of j-th size interval undergo breakage
$v_{i,j}(t)$	element of $\underline{\underline{V}}$ matrix, defined in Eq. (2.105)
$V(\bar{p})$	an unknown unimodal frequency function in $\bar{p}$
$\underline{\underline{V}}(t)$	a matrix defined in Eq. (2.105)
w	particle size
$\bar{w}$	pseudo- particle size
w^*	a weightage factor in Eq. (2.41)
W	particulate mass
x	particle size
$\bar{x}$	a characteristic parameter of the particle size distribution defined in Eq. (2.4)
x'	size modulus in Gate-Gaudin-Schumann distribution
x^*	particle size at which the selection function exhibits a maxima

x_0	maximum particle size
x_i	size of the openings of the lower sieve of i-th size interval
$\bar{x}_i$	defined in Eq. (2.75)
$\bar{\bar{x}}_i$	defined in Eq. (2.76)
$\bar{x}_f$	characteristic particle size of feed
$\bar{x}_p$	characteristic particle size of the product
$X_i(\mathcal{T})$	solution to Eq. (10.5)
$Y(\bar{u}, t)$	particle mass density function in pseudo-size $\bar{u}$ at time t
z	ratio of the size of the daughter particle to that of the parent particle
$Z(\epsilon/\epsilon_m)$	an unspecified function in Eq. (2.17)
α	a constant in Eq. (2.56)
α_1, α_2	constants
β	a constant in Eq. (2.57)
β_1, β_2, ..β_4	unknown constants in Eqs.(8.9) and (8.11)
$\bar{\beta}_1$, $\bar{\beta}_2$	unknown constants in Eq. (8.12)
δ	sieve size ratio, defined as the ratio of the size of the openings of the upper sieve of a size interval to that of the lower one
$\phi(b, a)$	a function defined in Eq. (7.30)
$\psi(x)$	an unspecified function of particle size in Eq. (7.19)

ϵ	some measure of fineness of the particulate solids
ϵ_m	limiting value of the fineness
$\epsilon_{i,j}$	a constant in Eq. (9.7)
π	a constant in Eq. (2.30) for the fraction of particles selected for breakage during are breakage cycle
$\pi_{i,j}$	a constant in Eq. (9.7)
$\theta(u,x,\bar{\bar{t}})$	mill function
$\bar{\theta}(v,x,\bar{\bar{t}})$	cumulative mill function
$\varphi(x)$	initial rate of formation of material finer than size x (zero-order production rate constant)
φ_i	initial rate of formation of material finer than size x_i(zero-order production rate constant)
Γ	gamma function
μ_r	r-th general moment of M(x,t)
ν	number of cycles of breakage
$\nu_r(t)$	r-th general moment of $Y(\bar{u},t)$
τ	scaled grinding time defined in Eq. (10.4)

SYNOPSIS

This investigation deals with mathematical modelling and identification of a batch ball mill grinding system. The study has been subdivided into : (1) a pseudo-similarity solution to the integro-differential equation of batch grinding , (2) exact mathematical definitions of the parameters in the discrete size grinding models in terms of the basic size continuous functions , (3) realizations of various specialized discrete size models , (4) analysis of some special grinding phenomena , (5) identification of the grinding system in terms of a mill function , (6) estimation of size discretized parameters , and (7) empirical correlations for variation of size-discretized rate parameters with particulate mass and ball size . A brief summary of the results is given below :

1. A Pseudo-similarity Solution to the Integro-differential Equation of Batch Grinding

A solution to the integro-differential equation of batch grinding in continuous size and time has been obtained using Kapur's similarity solution and standard mathematical transformation techniques . The equation describes the complete trajectory of the particle spectra in time , except for a small initial period . The non-self-preserving nature of the resulting size spectra has been illustrated by simulation studies. This

is the most general analytical solution to the grinding equation available at present .

2. Exact Mathematical Definitions of the Parameters in the Discrete Size Models

Starting from the fundamental integro-differential equation of batch grinding , exact mathematical difinitions of the parameters in the two types of discrete size models , currently being used , have been obtained in terms of the size continuous selection and breakage functions which can be defined unambiguously . It has been shown that , even when the system is inherently linear and time invariant, the size discretized parameters are , in general, implicit functions of grinding time. The time variability of the parameters is inherent in the discretization procedure itself as the parameters are now functions of the particle size distribution within the individual size intervals.

A new function, which plays a central role in the analysis of discrete size models, has been identified and named 'Selekage Function', $L(v,x) = S(v)\ \bar{B}(v,x)$. Its important properties have been delineated .

3. Realizations of Various Specialized Discrete Size Models

When the particles within a size interval are uniformly distributed in size, the most commonly used Bass model is realized . If the selekage function $L(v,x)$ can be reduced to $L_2(x)$,

the RSF model is obtained, which generates the Rosin-Rammler type or Rosin-Rammler distributions. Given that the selekage function is separable i.e., $L(v,x) = L_1(v)\ L_2(x)$, the SSF model is obtained. Model equations have been derived and the time dependence of the size discretized parameters S_i and $B_{i,j}$ has been investigated. The parameters S_i and $B_{i,j}$ are time -invariant in Bass and RSF models. The SSF model has time dependent S_i and time-invariant $B_{i,j}$. Time variation of S_i parameters and the inner breakage parameters $b_{i,i}$ in SSF model has been computed using Kapur's similarity solution. It has been shown that the inner-breakage parameter is not negligibly small even when the sieve size ratio is only $\sqrt[4]{2}$.

4. Analysis of Some Special Grinding Phenomena

The phenomena of first order disppearance kinetics, zero-order production of fines, Aliyavdin kinetics, difference similar (normalizable) breakage parameters, and variation of the S_i parameters with particle size according to a power law, have been investigated in light of the structure of the specialized discrete size models proposed in this study. It has been emphasized that a given discrete size model possesses concurrent properties none of which must be violated, otherwise serious inner-contradictions will develop.

5. Identification of the Grinding System in Terms of a Mill-Function

A new approach has been proposed for identification of

a linear, time-invariant, continuous size and discrete time grinding system, in which the size reduction occurs by impact fracture, chipping and abrasion. Batch ball mill grinding data for dolomite has been successfully used for illustration of the identification procedure . The role of the various grinding mechanisms has been discussed in light of the results obtained and, the principal advantages of the proposed identification scheme have been discussed.

6. Estimation of Size Discretized Parameters

Assuming that the size discretized parameters are sensibly constant over a narrow time interval, a scheme has been devised for exact back-calculation of all the size discretized parameters S_i and $B_{i,j}$ using Reid's solution to the batch grinding equation. The type of grinding data required is single size feeds ground for a fixed period of time . Next using the estimated mill function for dolomite, the required data has been generated over several successive intervals of fixed grinding time and, for first time, the time variation of the model parameters due to their dependence on the size distribution of the particles within the individual discrete size intervals, has been successfully computed.

7. Empirical Correlations for Variation of Size-Discretized Rate Parameters With Particulate Mass and Ball Size

Simple systematic graphical procedures for determination

of the empirical correlations for the variations in rate parameters S_i of the discretized batch grinding equation (the Bass model) with particulate mass and the size of the grinding media balls have been developed and successfully demonstrated for two actual grinding systems . The previously known correlations for rate parameters have been shown to be restricted cases of those obtained in the present study.

CHAPTER 1

INTRODUCTION

Comminution or size reduction of solid materials is one of the most important unit operations in many metallurgical , chemical and ceramic industries from both the technical , as well as , the economic points of view . In mineral dressing plants and cement industry , especially, each year hundreds of millions of tons of various solids are subjected to the size reduction operation . The typical values of power consumptions for the required size reduction of cement , lead-zinc ore , copper ore , carborundum and pigments are 50 , 18 , 12 , 42 , and 66 kWh/ton, respectively [1a] . In view of very large power consumption in these operations , there exists considerable economic incentive for improving the design , operation and control of the comminution systems .

In general three broad stages may be identified in a size reduction operation . In order of increasing average fineness of the feed material these are typically - (i) crushing in jaw , gyratory or cone crushers , (ii) grinding in rod mills , and (iii) ball milling . Depending on the initial

size of the feed , its hardness and the size of the product desired , in practice all the three steps may not be necessary in a particular size reduction operation . However , in many industries the size reduction operation invariably ends with ball milling , and whenever it is so , almost the entire final product is generated in the ball mills . This implies that the ball milling is the most crucial and important of all the three steps in the size reduction operation . The importance of the ball milling operation is further enhanced by the fact that invariably the power consumption in the ball mills is several times greater than that required for both the preceding size reduction stages combined . In view of these remarks it is not therefore surprising that the tumbling mills in general and ball mills in particular , have attracted the maximum attention of the comminutionists .

Till recently , much of the design and operation of the ball mills was based on trial and error experience . However , with growing realization of the limitations inherent in the empirical approach , there now exists considerable interest in the fundamental understanding and analysis of the various physical phenomena which operate in the ball milling process , e.g. , the rate and mode of

breakage of the particles , the size distributions of the resulting progeny particles and also the transport of the material through the mill . The goal of these efforts is that , in principle , it should be possible to formulate a complete quantitative description of the grinding process starting from the first principles in terms of only fundamental physical properties of the mill - material system and various operating variables such as the mill speed , the feed rate to the mill and per-cent water in feed etc . The process of ball mill grinding being physically quite complex, such a development , however , will take considerable more effort and time . In order to provide an alternative basis for design , analysis , simulation , optimization and control of the grinding process , the comminution engineers have in last two decades developed several simplified mathematical models for identification of the mill-material system . Depending on the extent to which various physical and process variables have been lumped together , these models exhibit considerable variations in the approach of formulation , the level of sophistication and the details of the process . Evidently, more sophisticated a model is , greater is the number of parameters which characterize the model , and therefore , the task of system identification

is correspondingly more difficult . On the other hand , over-simplified models have very limited utility from the point of view of process analysis in the actual practice .

The present study is based on an approach which is now widely employed by the research workers in the field of comminution . It is assumed that at any instant the number of particles of a given size selected for breakage is directly proportional to the total number of particles of that size . This is commonly known as the assumption of 'first order disappearance kinetics ' . The second assumption is that the grinding behaviour of the particles is not affected by the constitution of the particulate assembly . In view of these two assumptions the system model in continuous size and time variables is therefore , linear in the mass fractions of the particles of different size (i.e., in the mass frequency function) and unless the mill-material system or the operating conditions change with time , it is also inherently time invariant . The model incorporates two basic functions - (i) the selection function , which gives the rate of selection for breakage of particles of a given size , and (ii) the breakage distribution function , which gives the fraction of progeny particles finer than a particular size when the parent particle undergoes a single primary breakage event .

In view of the fact that the experimental size distribution data is available only in terms of discrete size fractions , size discretized form of the above model is more convenient and also widely used . However , the size discretized models currently employed by various schools have either been formulated without any reference to the more fundamental size continuous model and the two basic defining functions , or are based on such assumptions which are not valid in most practical cases . This arbitrariness has led to some controversy and , at the same time , has given rise to some interesting problems . First , many such discrete size models can be shown to lead to several inner contradictions , and not surprisingly have contributed to misinterpretation of numerous experimental data . Second , because the parameters of the size discretized model are erroneously considered to be completely uncorrelated and independent of each other , estimation of a very large number of parameters (for example 78 for 12 size intervals) becomes extremely involved in both the experimental as well as the computational efforts .

With these problems in view , following tasks have been attempted in the present study :

(i) An analytical pseudo-similarity solution to the 'fundamental' integro-differential equation of batch grinding

has been obtained .

(ii) Exact mathematical definitions of the parameters of discrete size models in terms of the two basic size continuous functions have been established using a formal mathematical procedure .

(iii) Various specialized discrete size models have been identified and the important characteristics of their parameters have been investigated .

(iv) Various special grinding phenomena have been investigated in light of the structure of the specialized discrete size models .

(v) A new approach has been proposed and successfully demonstrated for identification of the grinding system in continuous size and discrete time .

(vi) A scheme has been proposed for exact back - calculation of the parameters using single size feed grinding data . The time variation of model parameters due to their dependence on the particle size distribution within the discrete size intervals , has been computed .

(vii) Graphical procedures have been developed and demonstrated for determination of the empirical correlations for the variations in rate parameters with particulate mass and the size of the grinding media balls .

CHAPTER 2

LITERATURE REVIEW

In last 50 years or so considerable progress has been made in the development of mathematical models of the size reduction processes. For convenience in presentation, the extensive literature on this topic can be classified into four broad categories :

1. Models with a single time-variable parameter
2. Models based on the concept of discrete repetitive breakage steps
3. Size and time continuous particle population balance models
4. Size discretized time continuous particle population balance models

As mentioned in Chapter 1, the discrete size models in the last category are of more practical importance. Therefore, a detailed literature review of the parameter estimation techniques and empirical correlations for variation of these parameters with operating variables has also been included.

2.1 Models With A Single Time Variable Parameter

The earlier attempts for the analysis of size reduction processes were based on a description of the system dynamics in terms of only a single characteristic

parameter of the particle size distribution as a function of the energy received by the particles for breakage |1-9| .

Typical of such models is the one due to Walker et al. |1| who proposed the following relationship

$$\frac{dE}{d\bar{x}} = - \frac{C_1}{\bar{x}^r} \qquad (2.1)$$

where E is the total specific energy of breakage applied to the particles during the course of grinding, r and C_1 are constants for a given mill-material system, and $\bar{x}$ is the characteristic size of the particle size distribution defined as

$$\frac{1}{\bar{x}^{r-1}} = \frac{\sum_i (M_i / x_i^{r-1})}{\sum_i M_i} \qquad (2.2)$$

In defining $\bar{x}$ the entire population of particles has been divided into several size groups of narrow size interval width, each size interval being characterized by an average size x_i . M_i is the mass fraction of particles in the i-th size interval. When r=1, the model in Eq. (2.1) is in accordance with the Kick's law of comminution. If r is assigned a value 2, the model represents the Rittinger's law. Any other value for r should be regarded only as an empirical constant.

Due to a lack of proper appreciation of the correct definition of $\bar{x}$, as given in Eq. (2.2), Walker et al.'s

model has been subjected to considerable misinterpretation in the subsequent literature. Some researchers |5-7| have confused $\bar{x}$ for the size of an individual particle and others |2-4, 8-10| have assigned to it various arbitrary meanings.

It should be noted that Eq. (2.2) is not meaningful when r=1. As shown by Austin |11|, in this case correct definition of $\bar{x}$ in continuous size variable should be

$$\log (\bar{x}) = \int_0^{x_0} \log (x)\, M (x)\, dx \qquad (2.3)$$

where M(x) dx is the mass fraction of particulate charge in an infinitesimal size range x to x+dx , and x_0 is the maximum particle size. Similarly the size continuous version of Eq. (2.2) is

$$\frac{1}{\bar{x}^{r-1}} = \int_0^{x_0} \frac{1}{x^{r-1}}\, M(x)\, dx \;, \quad r \neq 1 \qquad (2.4)$$

or

$$\bar{x} = (\mu_{1-r})^{1/1-r} \;, \quad r \neq 1 \qquad (2.4a)$$

where μ_{1-r} is the (1-r)-th moment of the particle size distribution.

Bond |2,3| noted that the energy input increases as particle size decreases and postulated that the specific energy input is inversely proportional to a characteristic particle size raised to an exponent 1/2. The equation proposed by Bond is

$$E = C_2 \left[\frac{1}{\bar{x}_p^{0.5}} - \frac{1}{\bar{x}_f^{0.5}} \right] \qquad (2.5)$$

where subscripts p and f refer to the product and the feed, respectively, and C_2 is a mill-material constant. This principle he has stated as - ''the energy input is directly proportional to the new crack length produced''. The implied relationship between the characteristic size of the particle size distribution $\bar{x}$, which he defined arbitrarily as 80 per-cent -- by weight - passing size, and the crack length, however, does not seem to be evident.

Replacing the constant 0.5 in Eq. (2.5) by a variable parameter r_v , we get the Holmes' model |4| . Holmes called r_v as Kick's law deviation exponent. But in view of the difference in the definitions of $\bar{x}$ in Eqs. (2.3) or (2.4) and that assumed by Holmes (i.e. 80 per-cent passing size), this interpretation does not make any sense.

Charles |5| assumed that the particle size distribution of the ground product could be approximated by the Gate-Gaudin-Schumann distribution :

$$F(x) = \left(\frac{x}{x'} \right)^{m'} \qquad (2.6)$$

where F(x) is the fraction of material finer than size x, x' is the size modulus and m' is the distribution modulus. Combining Eq. (2.6) with an arbitrary version of Eq. (2.1) in which $\bar{x}$ is replaced by particle size x, Charles obtained

for large reduction ratios

$$E = \frac{C_3 m'}{(r-1)(m'-r+1)} (x')^{1-r} \qquad (2.7)$$

It should be pointed out that Eq. (2.1) has no meaning when $\bar{x}$ is replaced by x |11| . Accordingly, the derivation of the Charles' model is untenable.

Some workers have formulated their models in terms of the specific surface area 'Q' of the particles. Tanaka |8| pointed out that with increasing fineness of the material, there is progressive depletion of the flaws in the material and also the frictional losses increase. Accordingly, he proposed the following empirical equation

$$\frac{dQ}{dE} = C_4 (Q_\infty - Q) \qquad (2.8)$$

where Q is the specific surface area , Q_∞ is the limiting value of Q, and C_4 is a mill-material constant.

Harris |9| has preferred to use 'the net energy input to the charge' E' as the independent variable in his model for fine grinding operation in batch mills instead of 'the energy responsible for fracture inducing environment' E. He argued that as particles become finer dissipation of energy at the points of contact between particles and the inter particle pulp increases and, therefore, E is a decreasing function of E'. He assumed

$$\frac{dE}{dE'} = f(E') = \frac{1}{E'} \qquad (2.9)$$

Using the logistic growth hypothesis he proposed the following expression

$$\frac{1}{Q} \frac{dQ}{dE} = K - F(Q) \qquad (2.10)$$

where K, the potential growth factor, according to Harris, is a constant in conformity with the Kick's law. This statement, however, is incorrect. The expression for Q is

$$Q = C_5 \int_0^{x_0} \frac{1}{x} M(x)\, dx \qquad (2.11)$$

Combining Eq. (2.11) with Eq. (2.3) and substituting in Eq. (2.1) with $f=1$, it can easily be confirmed that the expression

$$\frac{1}{Q} \frac{dQ}{dE} = K$$

does not follow from the Kick's law.

The second term F(Q) on RHS in Eq. (2.10) has been termed as 'retardation factor' by Harris. F(Q) is an increasing function of Q and has been introduced to account for the increasing resistance to fracture as particles become smaller . The boundary condition on F(Q) is

$$F(Q_\infty) = K \qquad (2.12)$$

and the assumed functional form for F(Q) is

$$F(Q) = \frac{KQ}{Q_\infty} \qquad (2.13)$$

Combining Eqs. (2.9), (2.10) and (2.13) Harris obtained

$$\frac{dQ}{dE'} = \frac{KQ(1-Q/Q_\infty)}{E'} \tag{2.14}$$

which integrates to

$$\left(\frac{E'}{\bar{E}}\right)^K = \frac{Q/Q_\infty}{1-(Q/Q_\infty)} \tag{2.15}$$

where $\bar{E}$ is the energy to produce half the maximum surface area $Q_\infty/2$.

It should be pointed out that all the models discussed above have a common major drawback. The only energy input to the system which can be directly measured is the gross energy input E_g , which is given by

$$E_g = \int_0^t P(t)\, dt \tag{2.16}$$

where $P(t)$ is the power draught to the mill. However as the models have been formulated either in terms of E or E' , it is obvious that the validity of such models cannot be confirmed without making further assumptions about the relationships of E and E' with E_g .

Harris [12] has also proposed an alternative model now using grinding time as the independent variable. The model is descriptive and is based upon a detailed analysis of the energy utilization in tumbling mills and various grind limit mechanisms . Formally, the model can be represented by the following equation

$$\frac{d(\epsilon/\epsilon_m)}{dt} = Z\,(\epsilon/\epsilon_m) \qquad (2.17)$$

where ϵ is some measure of the fineness (e.g., specific surface area), ϵ_m is the limiting value of the fineness and Z is the product of five different unspecified functions . Substituting for Q, Q_∞ and E' in Eq. (2.14) by ϵ , ϵ_m and t, respectively, and using the following equation

$$\frac{\epsilon_m}{\epsilon} - 1 = \left(\frac{p}{t}\right)^{\bar{n}} \qquad (2.18)$$

which, according to Harris, provides best fit to many grinding data, the following equation was obtained

$$\frac{d(\epsilon/\epsilon_m)}{dt} = \frac{\bar{n}}{p}\left(\frac{\epsilon_m}{\epsilon}\right)^{\frac{1}{\bar{n}}-1}\left(1 - \frac{\epsilon}{\epsilon_m}\right)^{\frac{1}{\bar{n}}+1} \qquad (2.19)$$

where $\bar{n}$, p and ϵ_m are three model parameters. According to Harris consistency of Eq. (2.19) with Eq. (2.17) demonstrates that Eqs. (2.14) and (2.18) are compatible with the model proposed by him. This does not, however, prove that the model in Eq. (2.19) is also unique.

In fact, it will not be an exaggeration to say that all the models described above result in little more than the curve fitting equations. Another major drawback with these models is the lack of any information on time variation of the particle size distribution. In fact excluding the Charles' model, which is in any case highly arbitrary, none

of these models give an explicit expression from which the particle size distributions at various grinding times may be predicted. It should be pointed out that a single parameter cannot uniquely represent the particle size distribution unless the latter exhibits a similarity or self-preserving character, that is the trajectory of particle size spectra in a batch milling operation is described by a single expression for the frequency function involving only one time dependent parameter . Kapur |13| has presented a comprehensive derivation of all the best known single time variable parameter grinding models based on a similarity solution to the integro-differential equation of batch grinding |14| . This will be discussed in Chapter 2.3.3 .

2.2 Models Based On The Concept Of Discrete Repetitive Breakage Steps :

Several mathematical models |15-18, 20,21| have been proposed in which the breakage process has been conceived in terms of discrete repetitive breakage 'steps' or 'cycles' . The concept of breakage step or cycle is easy to apply in the case of repeated shatter tests or single passage comminution devices such as the cone mills. It is, however, difficult to extend this concept to the size reduction operation in tumbling mills where due to

continuous rotation of the charge and the grinding media it is not possible to isolate a distinct repetitive breakage step for the entire system. Nevertheless, these models have indirectly contributed quite significantly to the present state of development of the grinding kinetics and , therefore, a brief review will not be much out of place.

Brown |15| assumed that for size reduction of coal, one breakage cycle can be characterized by the following distribution function :

$$C(v,x) = \frac{1 - \exp(-x)}{1 - \exp(-v/\bar{m})} \qquad (2.20)$$

$$, \quad 0<x<v/\bar{m}, \quad \bar{m} \geq 1$$

where x is the size of daughter particles, v is the size of parent particles, and $\bar{m}$ is a model parameter for the grinding system . Let, $F_q(x)$ represent the fraction of material finer than size x after q cycles of breakage, then by mass balance :

$$F_q(x) = F_{q-1}(\bar{m}x) + \int_{v=\bar{m}x}^{x_0} C(v,x)\, d_v F_{q-1}(v) \qquad (2.21)$$

$$\text{where } d_v F_{q-1}(v) = \frac{d}{dv}\left[F_{q-1}(v)\right] dv \qquad (2.21a)$$

This model suffers from two major drawbacks - (i) it assumes that no particle can escape unbroken during a breakage cycle, which is not normally true for many grinding operations and (ii) the choice of the mathematical form for

the function C is quite arbitrary, which, moreover, in view of only a single adjustable parameter $\bar{m}$ does not provide adequate flexibility for adaptation to the changes in the process characteristics due to variation in the operating conditions and changes in the breakage characteristics of the charge. It is also evident that the model is simply a black-box representation of the process and, therefore, no insight can be obtained into the breakage phenomena.

Epstein [16] introduced a new concept - that of a 'breakage event', which consists of merely a single step or event of degradation in the breakage history of a particle. His model is based on two assumptions - (i) the probability of breakage of any particle during q-th step of the process is a constant p_q , $0 \leq p_q \leq 1$, independent of the size of the particle, the presence of other particles, and the number of steps that have occured prior to a given step, (ii) the distribution $\bar{B}(v,x)$ of the broken pieces obtained from the application of a single breakage event to a given particle of size v is independent of the size of the particle broken in the sense that the fraction by weight of material having size less than zv ($0 \leq z \leq 1$) arising from the breakage of a unit mass of size v is independent of v, i.e., $\bar{B}(v,x) = \bar{B}(x/v)$. Thus the composite distribution function arising from the primary breakage of a unit mass

of size v is given by

$$D_q(x/v) = p_q \bar{B}(x/v) + (1 - p_q) A(x/v) \qquad (2.22)$$

where

$$A(z) = 0 \quad , \quad 0 \leq z < 1$$

$$= 1 \quad , \quad z \geq 1$$

And, the overall size distribution of the particulate material $F_q(x)$ after q steps of breakage is related to $F_{q-1}(x)$ by the equation

$$F_q(x) = \int_0^{x_0} D_q(x/v)\, d_v F_{q-1}(v) \qquad (2.23)$$

Incorporation of the breakage probability concept and introduction of the concept of breakage event are two most striking features of the Epstein's model. The distribution function $\bar{B}$ which represents primary breakage of a particle, due to a single act of degradation, provides more meaningful basis for process description as compared to the function C(v,x) used by Brown, which represents the overall breakage process in which the repeated breakage of resulting progeny particles is also included. However, a priori assumption of size independence of breakage probability p and restriction on the primary breakage distribution function $\bar{B}$ should be criticized. Epstein showed that under these two conditions particle size distribution F tends to be logarithmico-normal asymptotically.

From practical view point this result is not of much utility, since normally materials are not ground to this extent. Apart from this, calculation of the product size distribution by repeated application of Eq. (2.23) is quite a cumbersome job, - especially when the number of breakage cycles is large.

With a view to overcome the problem of evaluation of repeated integrals, Broadbent and Callcott [17] proposed a practical size discretized matrix model as follows:

Let the entire particulate assembly be divided into n size intervals , where size interval i (i = 1, 2, ...n) is bounded by two adjacent standard sieves of mesh openings x_i and x_{i-1} ($\delta = x_{i-1}/x_i > 1$) . Then, for sufficiently large n, the integrals, encountered in the last two models, may be regarded as the limiting cases of matrix multiplication. For example, if $c_{i,j}$ is the proportion of a typical particle between x_{j-1} and x_j before breakage which falls between x_{i-1} and x_i after one breakage cycle (i > j), then the breakage process can be represented as

$$\underline{p} = \underline{\underline{C}}\,\underline{f} \qquad (2.24)$$

where $\underline{p}$ is the n x 1 frequency vector describing the size distribution of the product, $\underline{f}$ is the n x 1 size distribution vector for the feed, and $\underline{\underline{C}}$ is the n x n breakage matrix

which consists of n vectors representing the size distributions of the product due to breakage of each size interval, corresponding to one cycle of breakage. Since the size of particles cannot be increased by breakage , $c_{i,j} = 0$ for $i < j$ and hence, $\underline{\underline{C}}$ is a lower triangular matrix.

Furthermore, since in each size interval i only a certain fraction $p^*_{i,i}$ may be selected for breakage, the model equation becomes

$$\underline{p} = \underline{\underline{D}}\,\underline{f} \tag{2.25}$$

where

$$\underline{\underline{D}} = \underline{\underline{C}}\;\underline{\underline{P}}^* + (\underline{\underline{I}} - \underline{\underline{P}}^*) \tag{2.26}$$

In Eq. (2.26) $\underline{\underline{P}}^*$ is a diagonal matrix consisting of $p^*_{i,i}$ elements and $\underline{\underline{I}}$ is the identity matrix . It can be shown that , in general, the product of $\bar{\nu}$ cycles of breakage will be described by

$$\underline{p} = \underline{\underline{D}}^{\bar{\nu}}\;\underline{f} \tag{2.27}$$

As compared to n x $\bar{\nu}$ evaluations of integrals in Eq. (2.23), computation of $D^{\bar{\nu}}$ by repeated matrix multiplication is certainly more convenient. This is the advantage of size discretized matrix model over the size continuous ones discussed earlier.

It should, however, be pointed out that the model

in Eq. (2.25) should not be considered as the discrete analog of Epstein's model in Eq. (2.23). Equation (2.23) represents breakage product corresponding to a single event of primary breakage. On the other hand, the model in Eq. (2.25) is for one breakage cycle , in which there may be more than one primary breakage events i.e., the product of breakage due to one event may be subjected to repeated breakage during the course of a breakage cycle.

Broadbent and Callcott arbitrarily chose the following form for distribution function C :

$$C\ (v,x) = \frac{\left[1-\exp(-x/v)\right]}{\left[1-\exp(-1)\right]} \quad , \ 0<x\leq v \qquad (2.28)$$

and following equation was used for calculating the elements of the $\underline{\underline{C}}$ matrix

$$c_{i,j} = \frac{\exp(-x_i/x_j\delta^{0.5}) - \exp\ (-x_{i-1}/x_j\delta^{0.5})}{1-\exp\ (-1)} \qquad (2.29)$$

where it has been assumed that the breakage of particles in size interval j can be approximated by the function C(v,x) corresponding to $v = x_j\delta^{0.5}$, i.e., an intermediate size inbetween x_j and x_{j-1} . It was further assumed that the fraction selected for breakage is independent of the particle size i.e., $p^*_{i,i} = \pi$, a constant, for all i. The expression for $\underline{\underline{D}}$ matrix , therefore, is :

$$\underline{\underline{D}} = \pi \underline{\underline{C}} + (1 - \pi) \underline{\underline{I}} \qquad (2.30)$$

The value of π , the only unknown parameter in the model, was determined using the standard least squares method for fitting data. Though, for some grinding systems a satisfactory fit to the experimental data could be obtained, the value of π showed significant variations from cycle to cycle. While in most of the cases the value of π constantly decreased for successive cycles, in some cases random variations were also observed.

In order to explain the failure of the above model in many cases to provide an acceptable fit, - even for a single breakage cycle - , a new concept of 'severity of breakage ' was advanced by these authors |18| . They proposed that in certain cases the material selected for breakage may break more severely than that described by the assumed form for the function $C(v,x)$, and this can be taken into account by raising the matrix $\underline{\underline{C}}$ to a variable power 'e'. The modified expression for D matrix can be written as

$$\underline{\underline{D}} = \pi \underline{\underline{C}}^{e} + (1 - \pi) \underline{\underline{I}} \qquad (2.31)$$

A graphical curve-matching procedure was used for determination of the parameters e and π, which cannot be said to be very satisfactory.

Several comments can be made on the structure of the Broadbent and Callcott's model - (i) the choice of a particular form for the matrix function $\underline{\underline{C}}$ is quite arbitrary. It is highly unlikely that a wide variety of grinding systems under widely different operating conditions will conform to the same type of breakage pattern - differing only in the fraction selected for breakage . Even the modified form $\underline{\underline{C}}^e$ can generate only a fixed set of breakage patterns, (ii) it has been frequently observed that the probability of breakage varies considerably with the particle size . Hence, the assumption of size independence of $\underline{\underline{P}}^*$ matrix is also not valid in general (iii) the model cannot account for the variation of its parameters from cycle to cycle. It may be mentioned that Gupta and Kapur |19| successfully simulated some of the grinding systems studied by Broadbent and Callcott |17| using a fixed set of parameters for all breakage cycles . Thus even though the grinding system is invariant, Broadbent and Callcott's model gives a false notion that perhaps the grinding behaviour of particles is varying from cycle to cycle. In fact, it so happens that due to incorrect assumptions in the model, the least square estimates of the parameters are biased to the particular set of grinding data used and can serve satisfactorily only under the condition of nearly similar size

distributions. It is also obvious that the model is simply a data fitting scheme and is incapable of providing an unique identification of the grinding system. The utility of such models is, therefore , quite limited.

Broadbent and Callcott |17| , Callcott |20| and Lynch |21| have applied the above model to ball mills also . It has been proposed that a fixed period of grinding can be considered to be equivalent to a cycle of breakage, in which case the model is discrete in the time variable . On the other hand the fact remains that batch ball milling process is inherently continuous in time and therefore , a time discretized model cannot provide an adequate representation of the process . The need for time continuous model is felt more acutely in case of continuous mode of operation because the particles exhibit a distribution in their time of residence in the mill.

It should also be pointed out that the concept of probability of breakage is applicable only for an infinitesimal time period corresponding to the occurence of a single primary breakage event. If the grinding time corresponding to a single breakage cycle is large compared to the frequency of occurence of primary breakage events , (for example 2.5 min. in the work of Broadbent and Callcott) all the particles will not break in a uniform manner. Some particles

will have a greater chance for rebreakage and some less. In such a case , it is evident that a single function C(v,x) cannot be used to describe the size distribution of the daughter particles . This is a very serious drawback of the model.

2.3 Time and Size Continuous Particle Population Balance Models

2.3.1 Mathematical Models

Bass, in his outstanding paper |22| , presented the first derivation of a phenomenological time and size continuous population balance model of the grinding operation for a description of the time variation of the size distribution of the particulate assembly . The model is based on statistical considerations and assumes that over the entire size range of interest, population density of particles is sufficiently large such that meaningful satistical averages of various properties associated with the grinding system may be defined .

For model representation , two functions M(x,t) and G(v,x) were used, where M(x,t) dx represents the mass fraction* of the charge in the size range (x,x+dx) and G(v,x) dx represents the fractional rate of formation of

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* In the original work actual mass was used instead of mass fraction. However, the two representations are essentially equivalent and the basic model remains unaltered.

particles in the size range (x,x+dx) from primary breakage of the particles in the size range (v, v+dv) . Three assumptions were made regarding the nature of the size reduction process - (i) smaller grains do not associate to form larger ones i.e., there is pure disintegration of the particles and no agglomeration takes place. Hence , G(v,x) is defined only in the range $0 < x \leq v$ and is equal to zero for all $x > v$, (ii) process of size reduction is independent of the constitution of the particulate assembly and , hence, G(v,x) is invariant in grinding time , and (iii) the function G(v,x) is continuous in both the variables in the range $0 < x \leq v \leq x_0$ and so is the mass density function M(x,t) . Detailed arguments were given in support of this assumption, which are quite convincing and have been widely accepted |23 , 26| .

The derivation of the model equation proceeds as follows . Let us consider a part of the population in the size range (x, x+dx) equal to the mass fraction M(x,t) dx . At any instant of time t , the fraction leaving this size range is equal to the rate at which material finer than size x is being formed due to the breakage of particles in this infinitesimal size range , i.e.

$$M(x,t)\, dx \int_0^x G(x,w)\, dw$$

and , the fraction entering this size range due to breakage of coarser particles is

$$\int_{x}^{x_0} [G(v,x)\ dx\ ]\ M(v,t)\ dv$$

Hence, by mass balance it follows that

$$\frac{\partial [M(x,t)\ dx]}{\partial t} = -\ M(x,t)\ dx \int_{0}^{x} G(x,w)\ dw + \int_{x}^{x_0} [G(v,x)\ dx]\ M(v,t)\ dv \quad (2.32a)$$

which simplifies to

$$\frac{\partial M(x,t)}{\partial t} = -\ M(x,t) \int_{0}^{x} G(x,w)\ dw + \int_{x}^{x_0} G(v,x)\ M(v,t)\ dv \quad (2.32)$$

This is the fundamental partial integro-differential equation of batch grinding .

An alternate independent derivation of this equation was later given by Filippov [23] , which was based on a sophisticated probability theory approach . By similar mass balance considerations , Filippov obtained

$$\frac{\partial R(x,t)}{\partial t} = \int_{x}^{x_0} S(v)\ \bar{B}\ (v,x)\ d_v\ R\ (v,t) \quad (2.33)$$

and

$$\frac{\partial F(x,t)}{\partial t} = \int_{x}^{x_0} S(v)\ \bar{B}\ (v,x)\ d_v\ F(v,t) \quad (2.34)$$

where $R(x,t)$ is the statistical expected value of the fraction of particles coarser than size x, $S(x)\,dt$ is the probability of breakage of particles of size x in an infinitesimal time interval $(t, t+dt)$, $\bar{B}(v,x)$ is the mathematical expectation of the normalized distribution function, which describes the size distribution of the daughter particles arising from one splitting of the particles of size v, and $F(x,t)$ is the statistical expected value of the fraction of particles finer than size x. Some of the important assumptions made by Filippov are - (i) breakage of each particle is independent of its past and the presence of other particles (i.e., no particle-particle interactions), (ii) the function $S(x)$ depends only on particle size x and is bounded for the size interval under consideration (iii) the function $\bar{B}(v,x)$ depends only on v and x , and the sum of the mass of the daughter particles , obtained from one particle through one splitting is equal to the mass of the particle (i.e., $\bar{B}(v,v) = 1$) .

It can be seen that the Bass and Filippov models are time-invariant in nature . Furthermore, it can readily be shown that both are identical in respect of the system representation, too . By definition of the functions $M(x,t)$, $R(x,t)$ and $F(x,t)$, it follows that

$$M(x,t) = \frac{\partial F(x,t)}{\partial x} = -\frac{\partial R(x,t)}{\partial x} \qquad (2.35a)$$

and

$$M(x,t)\,dx = d_x\,F(x,t) = -\,d_x\,R(x,t) \qquad (2.35b)$$

Differentiating Eq. (2.34) with respect to **x** and making use of Eqs. (2.35a) and (2.35b) we obtain

$$\frac{\partial M(x,t)}{\partial t} = -\,S(x)\,M(x,t) - + \int_x^{x_0} S(v)\,\frac{\partial \bar{B}(v,x)}{\partial x}\,M(v,t)\,dv \qquad (2.36)$$

It clearly shows that Eqs. (2.32) and (2.36) are exactly similar in form. Further, by comparison of the Eqs. (2.32) and (2.36) , we obtain

$$S(x) = \int_0^x G(x,w)\,dw \qquad (2.37)$$

and

$$S(v)\,\frac{\partial \bar{B}(v,x)}{\partial x} = G(v,x) \qquad (2.38)$$

or, alternatively

$$\frac{\partial \bar{B}(v,x)}{\partial x} = \frac{G(v,x)}{\int_0^v G(v,w)\,dw} \qquad (2.39)$$

Equations (2.37) and (2.39) also establish that both functions $S(x)$ and $\bar{B}(v,x)$ in Filippov's model can be explicitly expressed in terms of $G(v,x)$, the single function used by Bass . Both the models are , hence , proved to be identical .

Interestingly , Gaudin and Meloy |24| and Gardner and Austin |25| also derived the above model independently .

Gardner and Austin derived Eq. (2.34) using functions S(x) and $\bar{B}(v,x)$ and named these functions as 'selection function' and 'breakage function', respectively. This nomenclature is also in accordance with the current practice . Gaudin and Meloy started with number balance on particles and a specialized functional form for $\partial\bar{B}(v,x)/\partial x$ |27| and obtained Eq. (2.36) by transformation to the mass basis. For convenience in writing , the density function form of the breakage function $\partial\bar{B}(v,x)/\partial x$ hereafter shall be denoted by B(v,x) .

It should also be mentioned that application of the above model to tumbling mills requires an additional assumption , namely , the breakage process is independent of the position of the particle in the mill |28| , i.e., the characteristic breakage event is same everywhere in the mill. However, if particles are fully mixed, such an assumption is not necessary , because in this case the average breakage event for particles of a given size remains invariant in time .

Lastly, the above model does not take into account grinding by abrasion or attrition .

2.3.2 Estimation of Model Functions and Model Verification in Continuous Size and Time

So far , there has not been even a single report in the literature where the aforementioned model has been verified

against the actual grinding data obtained from a physical system There have been two main difficulties in this respect . First, it has not been possible to determine the selection and breakage functions $S(x)$ and $\bar{B}(v,x)$ from purely theoretical considerations based only on the fundamental physical properties and dynamics of the mill-material system . Second, there are several physical limitations which preclude the direct determination of these functions using experimental methods. It is virtually impossible in practice to isolate the material of size x exactly . Equally difficult is the task of performing a grinding experiment with tumbling mills in which no particle undergoes more than one splitting . These difficulties do not permit us to take advantage of the well known methods of system analysis such as the tracer techniques . A brief account and criticism of the work done in this area is given below .

Bass |22| subdivided the size range $0<x\leq x_0$ into n size intervals of very narrow width such that it is reasonable to assume that the fraction of the product of breakage remaining in the original size interval is negligible . Under this condition he proposed a design of experiments which will permit determination of various parameters $v_{i,k}$, where

$$v_{i,k} = \frac{1}{x_{k-1} - x_k} \int_{x_k}^{x_{k-1}} \int_{x_i}^{x_{i-1}} G(v,x)\,dx\,dv \qquad (2.40)$$

and x_{i-1} and x_i are upper and lower size limits, respectively, for i-th size interval. It was further proposed that a generalized polynomial in v and x could be used to approximate the function G(v,x) and the coefficients of this polynomial could be determined by minimizing the following error function via the method of least squares

$$er = \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} \left[v_{ik} (x_{k-1} - x_k) - \int_{x_k}^{k_{k-1}} \int_{x_i}^{x_{i-1}} G(v,x)dx\, dv \right]^2$$

$$= + w^* G(x,x) \qquad (2.41)$$

where er is the error function and w^* is a weightage factor. In principle the method appears to be sound. However, there are several draw-backs which make it highly impracticable - (i) it requires that extremely narrow size intervals should be chosen . As to be shown later in Chapter 6 , size intervals as narrow as corresponding to the ratio $\delta = x_{i-1}/x_i = \sqrt[4]{2}$ cannot satisfy the condition of negligible inner-breakage . Therefore a large number of $nx(n-1)/2$ v's must be determined by performing a equally large number of experiments , (ii) due to the presence of unavoidable experimental errors in the measurement of the size distributions, $v_{i,j}$ values will also be in error. In absence of any knowledge of the magnitude of errors introduced in v's due to the experimental and computational procedures, the least squares method does not guarantee a real identification of the system, and

(iii) as to be shown later in Chapter 8 , grinding by abrasion cannot be neglected and , hence , Bass's assumption of $G(x,x)=0$ is not valid , in general* . The formulation of the error function in Eq. (2.41) is also, therefore , not quite valid . Lastly , no demonstration showing the potentialities of the above identification scheme using any real grinding data has been made .

Based on some theoretical considerations Gaudin and Meloy |24 , 27| derived the functional forms for both the functions $S(x)$ and $B(v,x)$. They did not, however , describe how the various contants in these expressions were to be determined for a given grinding system . Assigning arbitrary values to these parameters they computed size distribution curves for various grinding times and concluded that the nature of these curves was very much similar to those obtained in the actual practice . Again , the model was not tested against the grinding data from any real system .

Gardner and Austin |25| proposed a procedure whereby the size discretized analogs of selection and breakage functions could be used to determine the functions $S(x)$ and

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* It may be pointed out that this condition means that $B(x,x)=0$ for all x , which in view of various findings does not seem to be true .

$\bar{B}(v,x)$. These authors assumed that the selection function $S(x)$ for $x = (x_i + x_{i-1})/2$ is same as the selection parameter S_i * for i-th discrete size interval , and the cumulative breakage function $\bar{B}(v,x_i)$ for $v = (x_j + x_{j-1})/2$ is same as $\bar{B}_{i,j}$ * . It may be mentioned here only briefly that both S_i and $\bar{B}_{i,j}$ are determined by $S(x)$ and $\bar{B}(v,x)$ jointly , and , therefore , except in special cases , as to be discussed later in Chapters 5 and 6,it is impossible to find out $S(x)$ from S_i or $\bar{B}(v,x)$ from $\bar{B}_{i,j}$ alone . Hence , this procedure is completely arbitrary ~~and absurd~~ , and has no relevance whatsoever to the determination of the functions $S(x)$ and $\bar{B}(v,x)$.

In another paper |29| Gardner and Austin used smooth curves of $S(x)$ and $\bar{B}(v,x)$ to determine the size discretized parameters using the following two equations :

$$S_i = [\, S(x_i) + S(x_{i-1}) \,]/2 \qquad (2.42)$$

$$\bar{B}_{i,j} = [\, \bar{B}(x_i , x_j) + \bar{B}\,(x_i , x_{j-1})]/2 \qquad (2.42a)$$

It should be pointed out that in addition to the fact that Eqs. (2.42) and (2.42a)are ~~totally~~ incorrect ~~and absurd~~ , it hardly makes any sense to convert discrete

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* The definitions of these parameters are given in Chapter 2.4 .

parameters to continuous functions and then back convert continuous functions to discrete parameters . Moreover , it can be proved that the process of conversion used by them is not reversible .

In conclusion then it can be said that the population balance grinding model in continuous size and time has not been properly verified . However , in the next section we shall witness some indirect evidences in support of the model , which have been obtained using intutively appealing functional forms for the functions $S(x)$ and $\bar{B}(v,x)$ and comparing the experimental grinding data with the exact solution to the basic integrodifferential batch grinding equation .

2.3.3 Some Analytical Solutions to the Population Balance Batch Grinding Equation in Continuous Size and Time

The only general solution to Eq. (2.36) , the fundamental integro-differential equation of batch grinding , is due to Bass |22| , who obtained the following solution in terms of iterated kernels :

$$M(x,t) = f(x,t) \exp\left[-t \int_0^x G(x,w)\, dw\right] \qquad (2.43)$$

where

$$f(x,t) = M(x,0) + \int_x^{x_0} dv \int_0^t \sum_{i=1}^{\infty} H_i\,(x,t;v,t')\, M(v,0)\, dt' \qquad (2.44)$$

$$H_1 = - G'(v,x) \tag{2.45}$$

$$H_i (x,t;\ v,t') = \int_v^x dw \int_{t'}^{t} G(w,x)\ H_{i-1}(w,\bar{t};\ v,t')\ d\bar{t} \tag{2.46}$$

and

$$G'(v,x) = G(v,x)\ \exp\ [\ -\ t'(\ \int_0^v G(v,u)\ du - \int_0^x G(x,u)\ du)] \tag{2.47}$$

Because this solution is quite complicated and computationally quite time consuming , it has not found any practical use thus far .

Some simple solutions have been derived for specialized functional forms of the functions $S(x)$ and $\bar{B}(v,x)$. Gaudin and Meloy |24| showed that for

$$S(x) = A\ x \tag{2.48}$$

and

$$\bar{B}(v,x) = x/v \tag{2.49}$$

the solution to Eq. (2.36) in cumulative retained mode is

$$R(x,t) = R(x,0)\ \exp\ (-A\ x\ t\ ) \tag{2.50}$$

where A is a constant .

Austin et al. |30| presented a more general solution assuming the following restrictive relationship

$$S(v)\ \bar{B}(v,x) = A\ g(x) \tag{2.51}$$

Since , $\bar{B}(x,x) = 1$, it follows that

$$S(x) = A\ g\ (x) \tag{2.52}$$

and

$$\bar{B}(v,x) = \frac{g(x)}{g(v)} \qquad (2.53)$$

Furthermore , since $\bar{B}(v,x)$ is a monotonically non-decreasing function in x , g(x) should also satisfy this condition.. And , since $\bar{B}(v,0) = 0$, therefore , $g(0) = 0$. It follows that $S(0) = 0$ and S(x) is also a monotonically non-decreasing function of x . The solution to the grinding equation is

$$R(x,t) = R(x,0) \exp \left[- A\, g(x)\, t \right] \qquad (2.54)$$

or

$$M(x,t) = \exp \left[-A\, g(x)\, t \right] \left[M(x,0) + At \frac{dg(x)}{dx} \int_{x}^{x_0} M(v,t)dv \right] \qquad (2.55)$$

When g(x) = x, Eq. (2.54) reduces to the solution of Gaudin and Meloy |24| , $g(x) = - \ln \bar{g}(x)$ gives the solution of Harris |31| , and , for $g(x) = x^{\alpha}$, it gives the well known Rosin-Rammler size distribution equation . It should be mentioned that for many grinding systems studied in the past |17, 19, 32-36 | , Eq. (2.54) was found to hold reasonably satisfactorily . This may be taken as one indirect evidence in support of the grinding model.

Using the functional forms

$$S(x) = A\, x^{\alpha} \qquad (2.56)$$

$$\bar{B}(v,x) = \left(\frac{x}{v} \right)^{\beta} \qquad (2.57)$$

King |26| obtained the following analytical solution in

cumulative fraction finer mode

$$F(x,t) = \left(\frac{x}{x_o}\right)^{\beta} h(-\beta/\alpha\ ,\ \beta/\alpha\ ;\ 1;\ -A\,x_o^{\alpha}t\ ,\ -A\,x^{\alpha}t)$$
$$+ \int_{x}^{x_o} \left[\left(\frac{\beta}{x}F(w,0) - M(w,0)\right)\left(\frac{x}{w}\right)^{\beta} \exp\left(-Aw^{\alpha}t\right)\right.$$
$$\left.{}_1F_1\ (\beta/\alpha\ ,\ 1,\ (w^{\alpha} - x^{\alpha})\,A\,t)\right]\,dw \qquad (2.58)$$

where h (a, b, 1, x, v) is the generalized hypergeometric series in two variables and ${}_1F_1$ (a, 1, z) is the confluent hypergeometric function . Though , this restricted solution to the grinding equation is computationally more convenient as compared to the general solution obtained by Bass , however, generation of the values for the hypergeometric series and confluent hypergeometric function on digital computers is quite a problem . For hand calculations reference can be made to the tables |37| and the graphs given by King |26| . King gave an analytical solution for piecewise continuous selection function also. The selection function exhibits a maxima at the point of discontinuity x^* and is proportional to x^{α_1} for $x < x^*$ $(\alpha_1 > 0)$ and to $x^{-\alpha_2}$ for $x > x^*$ $(\alpha_2 > 0)$.

Under the restrictions assumed by King and given in Eqs. (2.56) and (2.57) , Kapur |14| obtained a very simple and useful similarity solution to the grinding equation

$$M(x,t) = \frac{C_o}{\mu_1(t)}\left[\frac{x}{\mu_1(t)}\right]^{\beta-1} \exp\left[-\frac{1}{\alpha h}\left(\frac{x}{\mu_1(t)}\right)^{\alpha}\right] \qquad (2.59)$$

where

$$C_0 = \frac{\alpha}{(\alpha h)^{\beta/\alpha}\ \Gamma(\beta/\alpha)} \qquad (2.60)$$

$$h = \frac{1}{\alpha}\left[\frac{\Gamma(\beta/\alpha)}{\Gamma[(\beta+1)/\alpha]}\right]^{\alpha} \qquad (2.61)$$

and Γ is the gamma function . The scale factor $\mu_1(t)$, the first moment of the distribution $M(x,t)$, varies with time in the following manner

$$\frac{d\,\mu_1(t)}{dt} = -\ Ah\left[\mu_1(t)\right]^{\alpha+1} \qquad (2.62)$$

Kapur |13-14| has shown that $\mu_1(t)$ is proportional to 80 per -cent passing size . It can be shown that it is also proportional to $\bar{x}$ as defined in Eq. (2.4) [Appendix 1] . Moreover , for a grinding period of moderate duration , it is frequently a fair approximation to assume that the power draught is constant |38| , and therefore , E_g , the gross energy input, is directly proportional to the grinding time t . Thus replacing μ_1 by $\bar{x}$, $(\alpha+1)$ by r and t by E_g , Eq. (2.62) can now be written as

$$\frac{dE_g}{d\bar{x}} = -\ \frac{C_6}{\bar{x}^{\,r}} \qquad (2.62a)$$

where C_6 is a proportionality constant , which incorporates the constants A and h of Eq. (2.62) . It will be seen that Eq. (2.62a) is similar to Eq. (2.1) , with the difference that

instead of E now E_g is the energy variable . As pointed out earlier in Chapter 2.1 , for a verification of the energy - size reduction laws it was invariably assumed that E or E' are directly proportional to E_g , an assumption for which no rational basis exists . On the other hand , Eq. (2.62a) , derived from considerations of particle population balance has in fact , exactly the same form in which all energy - size reduction laws have actually been verified . It is interesting to note that according to Kapur's derivation , the constant r is in no way related to any particular energy-size reduction law . Instead , it is function of the exponent α , which refers to altogether different phenomena - those which determine the size dependence of the probability of selection for breakage of particles . It should , hence , be very clear that in Eq. (2.62a) r = 2 corresponding to $\alpha = 1$ does not necessarily imply that Rittinger's law (i.e., $\Delta E \propto \Delta Q$) is valid (and similarly for r=1 and r=1.5) . On the contrary , two mill-material grinding systems exhibiting same value of α - the possibility of one obeying Kick's energy law and the other Rittinger's law cannot be ruled out .

Two important points emerge from above discussion . First , the extensive literature which is devoted to the verification of Eq. (2.62a) in one form or another , gives a wide support to the population balance model used by Kapur .

mill |39| were shown to be approximately self-preserving over a fairly wide range of the values of $\mu_1(t)$. This may be taken as another indirect evidence in support of the grinding model . Kapur |14| also indicated that the crude estimates of the parameters α and β could be obtained by simply matching the experimental similarity distributions with the standared theoretical ones for various values of α and β . Austin et al. |30| also suggested a graphical curve matching procedure for determination of these two parameters . Interestingly, neither Kapur nor Austin et al. estimated these parameters for any real grinding system . Therefore , the direct verification of the model has not been registered , thus far .

It may also be mentioned that under the constraints of Eqs. (2.56) and (2.57), exact solutions to the grinding equation have been derived by Filippov |23| and King |26| for the initial condition of an impulse situated at size x_0 . This result is of theoretical importance only , because in practice it is not possible to experiment with a material of exactly a single size .

2.3.4 Time Modified First Order Kinetics

Harris |38| and Kapur |13,14| have presented solutions to the fundamental integro-differential equation of batch grinding assuming specialized time dependent functional forms for the selection function . Harris assumed the following

restrictive forms for the two basic functions

$$S(x) = - At^{a} \ln \bar{g}(x) \qquad (2.63)$$

and

$$\bar{B}(v,x) = \frac{\ln \bar{g}(x)}{\ln \bar{g}(v)} \qquad (2.64)$$

where the constant 'a' serves the purpose of providing small systematic changes in the selection function with the grinding time t . The solution to the grinding equation Eq. (2.34) is

$$F(x,t) = 1 - \left[1-F(x,0) \right] \left[\bar{g}(x) \right]^{(\frac{A}{1+a})t^{1+a}} \qquad (2.65)$$

Harris noticed that when $a \neq 0$, Eq. (2.65) does not give consistent results . For example , if the grinding time interval t is divided into two subintervals of durations t_1 and t_2 , respectively (such that $t = t_1 + t_2$),and F(x,t) is formed by taking the product from first interval as the feed for the second interval , the result obtained is not the same as that obtained by directly substituting $t = t_1 + t_2$ in the Eq. (2.65) . Harris , therefore , inferred that the constant 'A' should in fact be a function of F(x,0) such that consistent results are obtained irrespective of the method of calculation used . He did not , however , make any attempt to find out this relationship .

Kapur |13,14| assumed the following form for the selection function

$$S(x) = A\, x^{\alpha}\, T(t) \qquad (2.66)$$

where $T(t)$ is an unspecified function in grinding time t, which represents the influence of particles' environment on the selection process. If the breakage function is governed by Eq. (2.57), then the first moment of the particle size distribution varies with grinding time as follows

$$\frac{d\mu_1(t)}{T(t)\,dt} = -A\,h\,[\mu_1(t)]^{\alpha+1} \qquad (2.67)$$

and a similarity solution to the integro-differential equation of batch grinding is given by Eqs. (2.59), (2.60), (2.61) and (2.67). In particular, for derivation of some energy - size reduction relationships Kapur |13,40| assumed the following form for function T

$$T(t) = 1/\mu_1(t)^{\bar{a}} \qquad (2.68)$$

where $\bar{a}$ is a constant

It should be pointed out that the grinding time itself does not directly influence the grinding operation in batch tumbling mills and, therefore, the grinding system as a whole cannot exhibit any explicit time dependence.

Any apparent time variability in the behaviour of the grinding system may be due to several reasons . One of the possible reasons , according to Austin et al. |41| , is due to the interaction between the distributions in strength of particles and in force applied by the grinding media . Mika et al. |42| and Kapur (13,40| have indicated the possibility of the influence of particle size distribution on the rate of breakage of particles . Also, Harris |38| has discussed how the proportion of active grinding sites may change with increasing fineness of the material . It will thus, be seen that the time variation of the breakage characteristics of the particles is determined by the initial conditions such as the strength distribution of particles at $t = 0$, the particle size distribution at $t = 0$, and several other similar possible factors |28| . Since , in different experiments, these initial conditions need not be the same , it is obvious that the time variability cannot be described as a unique function of time for all grinding experiments . In this respect , Kapur's approach should be less restricted since he used the first moment of particle size distribution to describe the time variation of the selection function . The 'a priori' assumption of the particular functional form in Eq. (2.68) may , however , be questioned .

Nakajima and Tanaka |43| derived the following approximate solution to the batch grinding equation under the restrictions given in Eqs. (2.56) and (2.57) :

$$F(x,t) \simeq 1 - [\,1 - F(x,0)\,] \exp\left[-(a_1 A x^{\alpha} t)^{a_2}\right] \qquad (2.69)$$

where a_1 and a_2 are constants determinable from β/α as described in their paper . These authors have tried to show that Eq. (2.69) , which is virtually the same as Harris' equation Eq. (2.65) for the following form of function $\bar{g}(x)$ |38|

$$\bar{g}(x) = \left[\,1 - \left(\frac{x}{x_0}\right)^{a_3}\right]^{a_4} \qquad (2.70)$$

can be derived without assuming a time dependent selection function . In a recent article |44| Austin et al. have pointed out that the derivation of Eq. (2.69) is suspect because the constant A is not a function of F(x,0) and , hence , it leads to the same paradox which was noted by Harris |38| in the case of Eq. (2.65) .

Before concluding this section a few comments are necessary regarding the need and justification for the use of time modified first order disappearance kinetics . It will be seen that if the selection and breakage functions are interrelated through a common function in the manner given in Eqs. (2.52) and (2.53) , then according to Eq. (2.54) a

ln ln [R(x,0)/R(x,t)] / ln t plot (Alyavdin - Weibull plot |45-47|) should result in a straight line of slope unity . There is a general opinion that as selection function and breakage function refer to altogether different phenomena , such an interrelationship between them is highly suspect , or even unacceptable |30,31| . It, therefore , follows that the deviations from a slope of unity as observed by Harris |45-46| are only natural and to be expected . But , at the same time , unless we are able to filter out the effects due to the violation of the above mentioned interrelationship between the two functions , there is hardly any need to assume a time dependent selection function in the first place. In fact, at present , it does not seem possible to analyze any inherent time dependent effects properly mainly because we do not have an adequate basis to obtain a precise knowledge of the selection and breakage functions .

In conclusion , the time modified first order kinetics as introduced by Harris through Eq. (2.64) has no physical basis whatsoever . Moreover , the graphical demonstrations of a wide applicability of Eq. (2.65) by Harris |45-46| should also be accepted with some reservations as there is an inherent tendency for linearization in the plotting procedure based on highly compressed log log versus log scales .

2.4 Size Discretized Time Continuous Particle Population Balance Models

As it can clearly be seen from the discussion in the previous section , because of the problems in the identification of the grinding system , as well as , the complexity of the general solution to the fundamental integro-differential equation of batch grinding , the size continuous model has found little use in practice . On the other hand , since the particle size distribution is invariably determined by sieve analysis and is available only in frequency form as mass fractions in discrete size intervals , there has existed a valid motivation for an appropriate practical size discretized model of batch grinding kinetics , first proposed by Sedlatschek and Bass |48| in the year 1953 . Since then , many authors |22, 28, 42, 48 - 54| have discussed the rational for and justification of the discrete size mathematical models formulated by them . A critical review of these models is presented below .

The notation used in this discussion is as follows . A discrete size interval is defined as the size range bounded by the size of the mesh openings of two adjacent sieves in a standard series , normally characterized by a constant sieve size ratio δ (> 1) . The entire size range of interest is divided into n size intervals . For i-th size

interval the size of the mesh openings of the upper sieve is denoted by x_{i-1} and that of the lower one by x_i . The upper - most size interval is represented by i = 1 and the maximum size limit of the particles by $x = x_0$. The mass fraction of particulate solids in i-th size interval at grinding time t is denoted by M_i (t) . In the literature wherever the actual mass of the particles has been used in the model formulation it has been replaced , for convenience and uniformity , by the mass fraction in the following discussion . This is permissible , because the basic treatment remains unaffected .

2.4.1 Mathematical Models

Based on statistical considerations Sedlatschek and Bass |48| proposed that the disappearance kinetics of particles in a small size interval i can be described by the following equation

$$\frac{d\, M_i\ (t)}{dt} = - v_{i,i}\ M_i\ (t) \qquad (2.71)$$

where $v_{i,i}$ is the specific velocity of breakage of the particles in the i-th size interval . This was confirmed by plotting ln M_i (t) vs . t , which resulted in straight lines with only slight deviations . It may be noted that the slope of this straight line is equal to the numerical value of the constant $v_{i,i}$. Further , a specific rate of appearance $v_{i,j}$

was defined as the fractional rate of formation of material in the i-th size interval due to the breakage of the material in the j-th size interval ($j < i$) . From mass balance it follows that

$$v_{j,j} = \sum_{i>j} v_{i,j} \qquad (2.72)$$

In general , the overall mass balance for any size interval i can be written as

$$\frac{d M_i (t)}{dt} = - v_{i,i} M_i (t) + \sum_{j=1}^{i-1} v_{i,j} M_j (t)$$

$$, i = 2, 3, \ldots \qquad (2.73)$$

From the simultaneous solution to the set of equations given by Eq. (2.73) and a series of grinding experiments carried out by them , Sedlatschek and Bass obtained the values of the coefficients $v_{i,j}$'s for alumina - ball mill grinding system, which was characterized by four size intervals . These values produced a satisfactory simulation of the actual grinding experiments .

In a later publication Bass |22| presented the derivation of the size discretized model in Eq. (2.73) starting from the fundamental integro-differential equation of batch grinding (Eq. 2.32) . The derivation was based on the following assumption

$$x_{i-1} - x_i << (x_{i-1} + x_i)/2 \qquad (2.74)$$

such that the following two conditions are satisfied - (i) the mass of particles remaining in the original size interval after undergoing breakage is negligible ; and (ii) in determination of the following average values

$$\int_{x_i}^{x_{i-1}} M(x,t)\, dx = M(\bar{x}_i, t)\,[x_{i-1} - x_i] \qquad (2.75)$$

$$\int_{x_j}^{x_{j-1}} dx \int_{x_i}^{x_{i-1}} G(v,x)\, M(v,t)\, dv = M(\bar{\bar{x}}_i, t) \int_{x_j}^{x_{j-1}} dx \int_{x_i}^{x_{i-1}} G(v,x)\, dv \qquad (2.76)$$

it is valid to write

$$M(\bar{x}_i, t) = M(\bar{\bar{x}}_i, t) \qquad (2.77)$$

The exact mathematical definition of the parameters $v_{i,j}$ is then given by Eq. (2.40) . In Chapter 2.3 the relevent criticism of this model in connexion with the first condition of Bass , has already been given , which need not be repeated here . A more practical and mathematically less stringent condition for time invariance of model parameters shall be discussed in the main body of this work .

Reid |49| proposed a 'practical batch grinding equation ' based on 'practical parameters' , which can be directly determined by experiments . He noted that those

particles , which after undergoing breakage remain in the original size interval , cannot be measured experimentally . Therefore , he considered it necessary to redefine the breakage event such that a particle is broken only if it is no longer retained in the original size interval . Reid pointed out that due to the new definition of breakage , the breakage function will vary with the screen ratio δ , and defined a modified breakage function $\bar{B}(w, x, \delta)$, which gives the size distribution of the products of primary breakage of material originally in the size range w to δw . Also, a set of breakage parameters $B_{i,j}$ were defined as - the weight fraction of the material broken finer than size x_j which reports to size interval i , when the material of size interval j undergoes primary breakage . Another set of parameters S_i , fractional rate of breakage for size interval i , was defined as

$$S_i = - dM_i(t) / M_i(t)\, dt \qquad (2.78)$$

Reid pointed out that S_i also incorporates the above definition of breakage . Further , he assumed it to be independent of time .

The particle population balance , when expressed in terms of the mass fraction and the parameters S_i and $B_{i,j}$, leads to the following kinetic expression for the batch grinding operation

$$\frac{dM_i(t)}{dt} = - S_i M_i(t) + \sum_{j=1}^{i-1} S_j B_{i,j} M_j(t) , \quad i = 2, 3, \ldots n \tag{2.79}$$

Reid mentioned that the time independent breakage parameter B reflects the inherent assumption that the size distribution within a size interval remains constant . If sieve size ratio δ is small , natural size distributions can be well approximated by straight lines over single size intervals and the assumption of a constant size distribution is valid . He , however , failed to realize that the same argument can be used to justify the use of time-invariant rate parameter S_i in Eq. (2.78) . Instead , he first went in a roundabout way to show that in general S_i is not independent of time (interestingly , even here his basic approach to the arguments is not correct) . Then, he invoked some experimental results on the decay of single size coal |25|, alumina |48| and quartz |50| ground in batch ball mills and pointed out that the time invariant rate parameter in Eq.(2.78) could provide a close approximation to the actual decay kinetics over the time range investigated , and hence, the assumption is valid .

The major criticism of Reid's work is that he formulated the size discretized model in isolation from the more fundamental size continuous model , and failed to provide

a meaningful interpretation to S_i and $B_{i,j}$ parameters in terms of the basic functions $S(x)$ and $\bar{B}(v,x)$, The lack of insight into the structure of these parameters prevented him from obtaining the exact mathematical conditions for time invariance of the two sets of parameters . As to be shown later in this work , these parameters are uniquely interrelated , but Reid's work gives a false impression that the two sets of parameters can assume independent values . Finally , it is very clear that both the definition of the parameter S_i as given in Eq. (2.78) , as well as, the definition of the breakage parameters $B_{i,j}$ given by Reid , do not refer to the particles remaining in the original size interval . It is then quite unnecessary to modify the definition of the breakage event . Moreover , it can be shown that in certain cases it will lead to confusing and misleading contradictions. For example , if chipping and abrasion are the dominant modes of grinding for particles in a particular size range , such that practically all the broken particles fall back in the original size interval , then , according to Reid's definition of breakage , even though the particles do not break , nonetheless the material gets ground , which is absurd . It must be emphasized that even those broken particles which do not break according to Reid's definition , do contribute to the values of the rate and breakage parameters .

A comparison of Eq. (2.71) with Eq. (2.78) and Eq. (2.73) with Eq. (2.79) reveals that the two models due to Bass and Reid are basically indentical , and it follows that

$$S_i \equiv v_{i,i} \tag{2.80}$$

and

$$S_j \, B_{i,j} \equiv v_{i,j} \tag{2.81}$$

We also define a cumulative breakage parameter $\bar{B}_{i,j}$ as follows :

$$\bar{B}_{i,j} = \sum_{k>i} B_{k,j} \tag{2.82}$$

The mathematical model of Mika et al. |42| , which can also be represented by Eq. (2.79) , is based on the assumption that the fraction of broken particles which remain in the original size interval is negligible i.e., $B_{i,i} = 0$. As to be shown later in this work (Chap. 6) , this assumption is grossly incorrect for practical sieve size ratios of $\sqrt{2}$ and $\sqrt[4]{2}$. Consequently , the definition of $B_{i,j}$, namely, the proportion of the primary breakage product of the j-th size interval which reports to the i-th size interval , is evidently incorrect . For the same reason , the use of term 'Selection Function ' for S_i parameters is also incorrect , because all the particles selected for breakage do not fall out of the original size interval . It should be emphasized

that the experimental measurements of the breakage parameter do not represent the actual size distribution of all the progeny particles , rather represent the distribution of only a fraction of such particles which fall out of the original size interval . Like Reid , these authors also did not make any attempt to identify the size discretized parameters in terms of the basic functions S(x) and B(v,x) . They assumed that the breakage parameters $B_{i,j}$ are time independent while , the time variability of S_i parameters was attributed to the influence of size distribution of the environment on the breakage probability of the particles .

Freeh et al. |51 , 52| made an ad-hoc stipulation that the ground material is fed forward to only next three or four smaller size intervals . This was perhaps motivated by the resulting convenience in the system identification due to a reduction in the number of unknown parameters . They called the parameters $B_{i,j}$ as 'inter-size flow coefficients', which in light of the discussion above seems to be a more appropriate term than the commonly used 'breakage function' . The arbitrary nature of this model should be criticized because mere data fitting is hardly equivalent to a realistic modelling .

Finally , it may be mentioned that the formal mathematical structure of the grinding models used by various investigators , such as Kelsall and co-workers |55 , 56| , Reid and Stewart |57| , Herbst and co-workers |58 - 60| , Klimpel and Austin |61| , Gardner and Sukanjanjtee |62,63| , Moreira et al . |64| , Kapur and Agrawal |65 - 67| and many others , is adequately represented by Eq. (2.79) . The differences are only in the terminology used and the definitions assigned to the parameters S_i and $B_{i,j}$, some of which have already been discussed earlier .

An important point for discussion is the precise relationship between the discrete parameters S_i and $B_{i,j}$, and the size continuous functions , $S(x)$ and $B(v,x)$. With the exception of a recent note published by Austin and Bhatia |68| , there seems to be a complete void in the grinding literature on this subject . In fact , a number of investigators , without appreciating the implications involved, have used these terms interchangeably |69 - 71| . One fails to understand from the literature that if $S(x) \propto x^{\alpha}$, then, how S_i is also proportional to x_i^{α} , and , similarly , if $\bar{B}(v,x) = (x/v)^{\beta}$, then how $\bar{B}_{i,j} = (x_i/x_j)^{\beta}$ |30 , 62 , 72 | . Examples can be quoted from the literature |62| where the functional forms used for discrete parameters are not consistent with the assumed functional forms for continuous

functions and , again , where the form chosen for selection parameters is inconsistent with the one chosen for breakage parameters |62 , 72 - 75| . In the latter case , it is impossible to find two appropriate mathematical functions for $S(x)$ and $B(v,x)$, which when used in the expressions for parameters S_i and $B_{i,j}$ shall give rise to the assumed expressions for the two sets of parameters . An erroneous procedure of interconversion by Gardner and Austin |25 - 29| was pointed out in a previous section |Chap. 2.3 | .

Assuming reasonably small size intervals , such that , over the size range $x_i \leq x \leq x_{i-1}$ the particle density $M(x,t)$ can be approximated by $M_i(t)/ (x_{i-1} - x_i)$, and from the very physical definition of the parameter S_i , i.e., the fractional rate at which material is falling below the bottom size of the interval i due to the breakage in single size i , Austin and Bhatia |68| obtained the following expression

$$S_i = \frac{\int_{x_i}^{x_{i-1}} S(v)\, \bar{B}(v,x_i)\, dv}{x_{i-1} - x_i} \qquad (2.83)$$

In the same fashion they obtained ;

$$S_j\, \bar{B}_{i,j} = \frac{\int_{x_j}^{x_{j-1}} S(v)\, \bar{B}(v,\, x_i)\, dv}{x_{j-1} - x_j} \qquad (2.84)$$

from which the expression for $\bar{B}_{i,j}$ follows immediately. In the present work the exact definitions of S_i and $B_{i,j}$ parameters have been derived in Chapter 5 , Eqs. (2.83) and (2.84) are shown to be a specialized case of the exact relationships.

Some investigators |53-54| have used a different form for representation of the discrete size grinding model. The grinding system is characterized by two new sets of parameters- the selection parameters s_i and the distribution parameters $b_{i,j}$, which by definition are exact discrete size analogs of size continuous selection and breakage functions S(x) and B(v,x) . The selection parameter s_i is defined as the fractional rate at which the material in size interval i gets selected for breakage . The distribution parameter $b_{i,j}$ represents the fraction of <u>all</u> material selected for breakage in the size interval j , which directly reports to the size interval i without rebreakage . Unlike the breakage parameters $B_{i,j}$, which refer to only those particles which <u>break out</u> of the size interval j , the distribution parameters $b_{i,j}$ give the true discrete size breakage distribution and therefore , $b_{j,j}$ is also a valid member of the set of the distribution parameters . The equation describing the

the disappearance kinetics of a single size fraction can now be written as

$$\frac{dM_i(t)}{dt} = - s_i(t)\, M_i(t) + s_i(t)\, b_{i,i}(t)\, M_i(t) \qquad (2.85)$$

And, the batch grinding equation is

$$\frac{dM_i(t)}{dt} = - s_i(t)\, M_i(t) + \sum_{j=1}^{i} s_j(t)\, b_{i,j}(t)\, M_j(t) \quad , \; i = 1,2,..n \qquad (2.86)$$

It should be noted that in contrast to Eq. (2.79) the upper limit of summation is now i instead of i - 1 . A comparison of Eq. (2.78) with Eq. (2.85) and Eq. (2.79) with Eq.(2.86) gives

$$S_i(t) = s_i(t)\, (1 - b_{i,i}(t)) \qquad (2.87)$$

and

$$S_j(t)\, B_{i,j}(t) = s_j(t)\, b_{i,j}(t) \qquad (2.88)$$

From Eqs. (2.87) and (2.88) it follows that

$$B_{i,j}(t) = \frac{b_{i,j}(t)}{1-b_{j,j}(t)} \qquad (2.89)$$

It will be seen that s_j , $b_{j,j}$ and $b_{i,j}$ cannot be explicitly expressed in terms of S_j and $B_{i,j}$ alone . Hence, only one way conversion given in Eqs. (2.87) and (2.89) is possible . It should be pointed out that from the practical

point of view there is hardly any reason to prefer Eq.(2.86) over Eq. (2.79) . This is because s_i and $b_{i,i}$ can always be lumped into one parameter S_i and the product $s_j\ b_{i,j}$ is also replaceable by $S_j\ B_{i,j}$. In other words, since $b_{i,i}$ is not required for the formulation of the model separately , it is unnecessary to have n more model parameters . At the same time , it should be admitted that the parameters s_j , $b_{j,j}$ and $b_{i,j}$ provide more insight into the breakage behaviour of the particles than the rate and breakage parameters S_j and $B_{i,j}$. From the point of view of theoretical analysis, therefore, the system model representation in Eq. (2.86) may prove to be more useful.

Olsen |53| and Whiten |54| have derived analytical expressions for s_i and $b_{i,j}$ in terms of the functions $S(x)$ and $B(v,x)$. Olsen rightly obtained the following expression for s_i .

$$s_i(t) = \frac{1}{M_i(t)} \int_{x_i}^{x_{i-1}} M(v,t)\ S(v)\ dv \qquad (2.90)$$

He assumed that $\bar{B}(v,x) = (x/v)^{\beta}$ and erroneously concluded that it implied $b_{i,j} = b_{i-j}$. His expressions for the distribution parameters are :

$$b_{i,i} = 1 - (\delta)^{\beta} \qquad (2.91)$$

$$b_{i,j} = (\delta)^{\beta(i-j)} \left[1-(\delta)^{\beta}\right] \qquad (2.92)$$

The relationships given by Whiten are :

$$s_i = \frac{\int_{x_i}^{x_{i-1}} S(v)\, dv}{x_{i-1} - x_i} \qquad (2.93)$$

$$b_{i,j} = \frac{\int_{x_j}^{x_{j-1}} \int_{x_i}^{x_{i-1}} B(v,x)\, dx\, dv}{x_{j-1} - x_j} \qquad (2.94)$$

It will be shown later in Chapter 5 that Eqs.(2.91) and (2.92) as well as Eqs. (2.94) and (2.93) all are incorrect . Moreover, we shall show that these parameters cannot be assigned unique values , because they are functions of the particle size distribution with the discrete size intervals.

Henceforth, we shall refer to grinding model in Eq. (2.79) as model 'A' and the one in Eq. (2.86) as model 'B'.

2.4.2 Analytical Solutions to the Size Discretized Batch Grinding Equations

Reid |49| was first to obtain an analytical solution to the set of n differential equations given by Eq. (2.79) . The solution for un-equal rate parameters is as follows :

$$M_i(t) = \sum_{j=1}^{i} h_{i,j} \exp(-S_j t) \qquad (2.95)$$

where

$$h_{i,j} = \sum_{k=j}^{i-1} \frac{S_k \, B_{i,k} \, h_{k,j}}{S_i - S_j} \quad , \quad k \neq i \tag{2.96}$$

$$h_{i,i} = M_i(0) - \sum_{k=1}^{i-1} h_{i,k} \quad , \quad i \neq 1 \tag{2.97}$$

and $$h_{1,1} = M_1(0) \tag{2.98}$$

In case two or more size intervals may exhibit same value of the rate parameter, i.e., $S_i = S_j$, then as shown by Olsen |53a| , Eqs. (2.96) and (2.97) should be modified as

$$h_{i,j} = \sum_{k=j}^{i-1} S_k \, B_{i,k} \, h_{k,j} \tag{2.99}$$

and

$$h_{i,i} = M_i(0) - \sum_{\substack{k=1 \\ k \neq j}}^{i-1} h_{i,k} \quad , \quad i \neq 1 \tag{2.100}$$

The set of n simultaneous differential equation in Eq. (2.79) is conveniently represented by a matrix equation

$$\frac{d\underline{M}(t)}{dt} = - \left| \underline{\underline{I}} - \underline{\underline{B}} \right| \underline{\underline{S}} \; \underline{M}(t) \tag{2.101}$$

where $\underline{M}(t)$ is a n x 1 vector of mass fractions $M_i(t)$, $\underline{\underline{B}}$ is a n x n lower triangular matrix of breakage parameters $B_{i,j}$ and $\underline{\underline{S}}$ is a diagonal matrix of rate parameters S_i . A matrix solution to Eq. (2.101) due to Herbst and Mika |76| is

$$\underline{M}(t) = \underline{\underline{N}} \; \underline{\underline{V}}(t) \; \underline{\underline{N}}^{-1} \; \underline{M}(0) \tag{2.102}$$

or

$$\underline{M}(t) = \underline{\underline{O}}(t)\,\underline{M}(0) \qquad (2.103)$$

where the elements of matrices $\underline{\underline{N}}$ and $\underline{\underline{V}}$ are , respectively

$$n_{i,j} = \begin{cases} 0 & i < j \\ 1 & i = j \\ \sum_{k=j}^{i-1} \dfrac{B_{i,k}\, S_k}{S_i - S_j}\, n_{k,j} & i > j \end{cases} \qquad (2.104)$$

$$v_{i,j}(t) = \begin{cases} \exp\,(-S_i t) & i = j \\ 0 & i \neq j \end{cases} \qquad (2.105)$$

The solution in Eq. (2.102) in the form of an input-output relationship is of great importance . The elemtnt $o_{i,j}$ of the input-output matrix $\underline{\underline{O}}(t)$ directly gives us the fraction of matrial originally in size interval j which after breakage for time period t is found in size interval i . It is interesting to compare the matrix $\underline{\underline{O}}$ with Callcott's mill matrix $\underline{\underline{D}}$. Needless to say that $\underline{\underline{O}}$ being explicitly defined in terms of physically identificable quantities S_j and $B_{i,j}$, and being a continuous function of time , is more meaningful and useful than $\underline{\underline{D}}$.

It can be seen that with the help of Eqs. (2.87) and 2.89) , the solutions discussed above can be easily extended or batch grinding model 'B' also.

With a view to obtaining simpler analytical solutions, apur |65-67| has presented a rational approach for a

systematic reduction of the grinding equation Eq. (2.79), which in the limit results in the well known Rosin-Rammler type grinding kinetics. Kapur |65| has shown that Eq.(2.79) in the cumulative retained mode can be written as

$$\frac{dR_i(t)}{dt} = - S_i R_i(t) + \sum_{j=1}^{i-1} R_j(t) \left[S_{j+1} \bar{B}_{i,j+1} - S_j \bar{B}_{i,j}\right] \tag{2.106}$$

where

$$R_i(t) = \sum_{j=1}^{i} M_j(t) \tag{2.107}$$

and

$$M_i(t) = R_i(t) - R_{i-1}(t) \tag{2.108}$$

For the case when

$$S_j \bar{B}_{i,j} = \varphi_i \tag{2.109}$$

where φ_i is some function of size index i only |58 , 77| , Eq. (2.106) reduces to

$$\frac{dR_i(t)}{dt} = - S_i R_i(t) \tag{2.110}$$

whose solution is

$$R_i(t) = R_i(0) \exp(- S_i t) \tag{2.111}$$

The condition under which Eq. (2.109) is valid shall be discussed later in Chapter 5. It may be noted that when

$S_i = A\, x_i^{\alpha}$, Eq. (2.111) reduces to the Rosin-Rammler size distribution equation.

Of several approximate solutions presented by Kapur |65-67| , mention should be made of at least one of them, which is quite important from the point of view of identification of the grinding system. Dividing Eq. (2.106) by $R_i(t)$ on both the sides and using two successive interations in Cauchy - Picard method of integration of differential equations |77| , Kapur |56| obtained

$$\ln \frac{R_i(t)}{R_i(0)} = - S_i t + \sum_{j=1}^{i-1} \left[\left(S_{j+1} \bar{B}_{i,j+1} - S_j \bar{B}_{i,j} \right) \times \left(\frac{\exp (G_j - G_i)\, t - 1}{(G_j - G_i)} \right) \frac{R_j(0)}{R_i(0)} \right] \qquad (2.112)$$

where

$$G_i = - S_i + \sum_{j=1}^{i-1} \left[(S_{j+1} \bar{B}_{i,j+1} - S_j \bar{B}_{i,j}) \left(\frac{R_j(0)}{R_i(0)} \right) \right] \qquad (2.113)$$

Further discussion on the utility of this solution will be given in the next section .

It will be shown in Chapter 5 that in general , the size discretized parameters are functions of the particle size distribution within the discrete size intervals and

therefore, in the strict sense, above mentioned solutions, which are based on time - independent parameters, are not valid . Only in those cases, where over the time period of interest all the parameters can be assumed to be reasonably constant, these solutions can find any use .

2.4.3 Estimation of Time Invariant Size Discretized Parameters

The parameter estimation techniques currently being used by various schools can be broadly classified as - (i) direct determination from the experimental data , (ii) analog simulation of the grinding system and curve fitting to the experimental data, (iii) minimization of an error function formulated in terms of experimental mass fraction data and the predicted results obtained using the analytical solution to the size discretized batch grinding equation , and (iv) back calculation using the experimental data and the analytical solution to themodel equation.

The most popular method for direct determination of the rate parameters is to grind single size feeds and plot ln $M_i(t)$ vs. t for the feed size j . Should the resulting curve come out to be a straight line, the slope of the line gives the value of the rate parameter S_j . If the same

experiment is carried out for a small duration of time, the breakage parameters $B_{i,j}$ are obtained by forming the ratios of the mass of material that appears in any lower size i to the total mass that is ground out of the feed size interval j . It should be pointed out that accurate determinations of breakage parameters is not possible when this method is employed. This is because in an ideal case the experiment should be performed for an infinitesimal time period such that no particle gets a chance for rebreakage . In practice, the duration of experiment has to be finite and, moreover, it should be sufficiently large so that measurable quantities of material are formed in the lower size intervals . Gardner and Sukanjnajtee |62| proposed that 10 per-cent of the material may be allowed to break out of the feed size interval . Klimpel and Austin |61| and Moreira et al. |64| used 1 minute grinding time which corresponded to approximately 20 per-cent reduction in the feed size material . Obviously , the parameter estimates obtained by these authors are very approximate . Austin and Luckie |78| have suggested that several experiments may be performed for different grinding times and the results so obtained may be extrapolated to zero time . They, however, admit that the procedure is very tedious

and time consuming , and therefore, it is not much promising .

Another variation of this method is to tagg the material in the size interval of interest with a tracer, which is commonly a radioactive tracer |25, 62, 63, 64| . This method has the advantage that the parameter estimates so obtained are representative of the actual conditions which we desire to simulate . In other words, if the size distribution of the environment has any significant effect on the grinding behaviour of the particles, this effect is also taken into account in this procedure . However , since the starting feed is not a single size feed but a feed having a natural size distribution of interest, the proportion of the tagged material may be quite small in some size fractions. Hence, the uncertainties in the measurement of broken tagged material which reports to the fine size intervals, would be relatively large and correspondingly the estimates of the breakage parameters comparatively less accurate .

Freeh et al . |52| and Reid and Stewart |57| used analogue , curve - fitting technique to determine the 'best fit' to the experimental data . The set of n simultaneous differential equations given by Eq. (2.79) were programmed for solution on an analogue computer and the values of the parameters were determined by trial and error

adjustment of the relevant potentiometer knobs. To fit the data for n-th size interval (where n is usually 10-12), it would require to adjust n potentiometer knobs simultaneously . Obviously, for n large (say n > 6) this procedure is not practically feasible . Perhaps , for this reason , in both the studies some simplifying assumptions were made to reduce the number of parameters. Freeh et al. assumed that the ground material is fed forward to only next two or three size intervals. Hence, only three or four potentiometers need be adjusted for size intervals corresponding to size index i > 4 . Reid and Stewart assumed that the breakage parameters are difference - similar (normalizable) i.e. , $B_{i,j} = B_{i-j}$. Hence, only two potentiometer knobs were required to be adjusted for each size interval . The simplifying assumption made by Freeh et al. is obviously quite arbitrary and has no physical justification , whatsoever . It will be shown in Chapter 7 that the time invariant difference - similar breakage parameter necessarily requires that the rate parameters conform to $S_i = A\, x_i^{\alpha}$. Many examples can be cited from the literature |55,61,71,75,79| which show that the rate parameters exhibit a maxima at some intermediate size interval . Hence, the assumption of difference - similar

breakage parameter is not valid in the general case . A lack of appreciation of this point has led to inner contradictions in many papers |61,63,64,79| including the work of Reid and Stewart . It can thus be seen that in addition to the problem of simultaneous adjustment of a large number of n knobs and, therefore, the possibility of introducing a high uncertainty in the estimates of those parameters to which the solution to the model equation is not quite sensitive, it also requires that a large number (at least 2n) of experiments should be performed in order that meaningful estimates may be obtained in the presence of unavoidable noise in the experimental measurements . In conclusion, from the point of view of the analysis of grinding systems which do not exhibit difference-similar breakage parameters , the method of analog simulation is unsuitable and impractical.

Klimpel and Austin |61| and Moreira et al. |64| used optimization methods to determine the values of the rate parameters . The breakage parameters were obtained using the direct method described above. The criterion chosen to represent the goodness of match was minimization of the summed squarred error in values of calculated and experimental mass fractions :

$$Er = \sum_{i=1}^{n} \left[M_i(t) - \hat{M}_i(t) \right]^2$$

where Er is the error function and $M_i(t)$ and $\hat{M}_i(t)$ are predicted and experimental mass fractions , respectively. Incidentally, Olsen |53a| has also used the same approach for determination of the selection and distribution parameters. Klimpel and Austin preferred to use a polynomial in the arithmetic mean of upper and lower sieve sizes of the i-th size interval to describe the size variation of the rate parameters and determined the rate parameters in terms of the coefficients of the polynomial . This method has the advantage that the parameters can be estimated in terms of a fewer number of constants and coefficients. However, in order that meaningful parameter estimates may be obtained, it is necessary to ascertain that the functional forms chosen for the two set of parameters are consistent with their basic definition . It will be shown in Chapter 6 that the time independent rate and breakage parameters are uniquely interrelated through the functions $S(x)$ and $\bar{B}(v,x)$ and therefore, the functional forms for the two sets of parameters cannot be chosen arbitrarily and independently. For example, Klimpel and Austin |61| assumed the breakage parameters to be difference - similar . In this case , for the reasons mentioned earlier , it was then necessary to assume $S_i = A\, x_i^{\alpha}$ instead of a polynomial in x_i . In fact,

the rate parameters estimated by Klimpel and Austin in combination with the assumed difference similar breakage parameters form a set, which is not consistent .

Assuming that the relationship in Eq. (2.109) is valid, Austin and Luckie |78| showed that the feed size rate parameter S_1 and the breakage parameters $B_{i,1}$ can be obtained from the following exact expressions ;

$$S_1 = - \frac{1}{t} \ln \frac{R_1(t)}{R_1(0)} \qquad (2.114)$$

$$B_{i,1} = \frac{\ln [R_i(t) / R_i(0)]}{\ln [R_1(t) / R_1(0)]} \qquad (2.115)$$

Herbst and Fuerstenau |58| noted that when the relationship given in Eq. (2.109) is valid, the rate of production of fines is constant for short grinding times and in particular

$$\frac{dF_i(t)}{dt} = \wp_i \qquad (2.116)$$

where $F_i(t)$ is the cumulative fraction finer than size x_i at time t, i.e.

$$F_i(t) = \sum_{j=i+1}^{\infty} M_j(t) \qquad (2.117)$$

Using the experimentally determined values of S_1 and ϱ_i and assuming that $\varrho_i \alpha \bar{x}_i^{\alpha}$, they derived following expressions for S_j and $\bar{B}_{i,j}$:

$$S_j = S_1 \left(\frac{\sqrt{x_j \, x_{j+1}}}{\sqrt{x_1 \, x_2}} \right)^{\alpha} \qquad (2.118)$$

$$\bar{B}_{i,j} = \frac{k_0}{S_1} \left(\frac{x_i}{\sqrt{x_j \, x_{j+1}}} \right)^{\alpha} \qquad (2.119)$$

where k_0 and α are system constants determinable from measured ϱ_i values . By substituting j for i in Eq. (2.109) and noting that by definition $\bar{B}_{j,j} = 1$, it can be seen that

$$S_j = \varrho_j \qquad (2.120)$$

Also, it can be readily shown from Eqs. (2.109) and (2.119) that

$$\bar{B}_{i,1} = \varrho_i / S_1 \qquad (2.121)$$

and

$$\bar{B}_{i,j} = \frac{\bar{B}_{i,1} \; S_1}{S_j} \qquad (2.122)$$

It can thus be seen that Eqs. (2.120) and (2.122) provide a much more convenient way for obtaining the parameters S_j and $\bar{B}_{i,j}$ than what has been suggested by Herbst and Fuerstenau . In fact unnecessary complications in their

work arose because they failed to realize the identity in Eq. (2.120) .

It will be shown in Chapter 7 that the condition in Eq. (2.109) does not necessarily imply that $\varrho_i \alpha x_i^{\alpha}$. In view of this fact, Eqs. (2.120) and (2.122) are also more general than those suggested by Herbst and Fuerstenau.

In addition to the above two criticisms, it should be pointed out that the condition $S_j \bar{B}_{i,j} = \varrho_i$ necessarily implies Rosin - Rammler type grinding kinetics . From Eq. (2.111) it follows that ln $R_i(t)$ vs. t plots should be straight lines in this case. The data for which the above method has been applied by Herbst and Fuerstenau does not exhibit this behaviour as can be seen from Figure 7 in their paper |58| . Consequently, their estimation of the parameters is invalid.

Austin and Luckie |78| have presented a back-calculation method, which they have applied for determination of the non-difference-similar (non-normalizable) breakage parameters |75 , 80| . They made two assumptions - (i) for short grinding periods, amount broken to less than size i from breakage of material in size interval j can be approximated by $\bar{B}_{i,j} S_j t [M_j(0) + M_j(t)] / 2$ and (ii) while determining $\bar{B}_{i,j}$ corresponding to a fixed j, $\bar{B}_{i,k}(k>j)$

can be approximated by $\bar{B}_{i+j-k,j}$ i.e., the assumption of a difference - similar breakage parameter is approximately valid in this case . Surprisingly, even though the method was proposed for determination of non-difference-similar breakage parameters, for testing the method they used exact numerically, generated data based on only difference -similar breakage parameters |80| . Consequently, it is doubtful if the method will serve the purpose for which it has actually been proposed. Moreover, it should be pointed out that the back-calculation methods are extremely sensitive to statistical fluctuations in experimental data |49,62|. Gardner and Sukanjnajtee |62| have shown using synthetic data with controlled random error, that even if the exact estimates of breakage parameters are available, the error in the estimated values of rate parameters for lower size intervals could be as high as 25-50 per-cent .

Austin and Bhatia |68| and Austin and Luckie |75| have proposed some empirical expressions for parameters S_i and $\bar{B}_{i,j}$ which involve five unknown constants. They have also described approximate graphical procedures to determine these constants, which are not very satisfactory.

Finally, mention may also be made of the approximate identification procedure suggested by Kapur |66 - 67| . The

method has the advantage that the number of parameters is reduced from n (n+1)/2 to only n |66| or 2 n |67| . Moreover, two grinding experiments with distributed feed are adequate for identification of the grinding system . As pointed out earlier, the size discretized parameters are inherently time-dependent. Hence, if time invariant approximate values only are to be used, Kapur's model should be adequate for at least routine simulation purposes .

We have thus seen that all work in the area of parameter estimation has been based on simplified models. In addition to the several restrictive assumptions pointed out earlier, parameters have invariably been assumed to be time invariant . In Chapters 8 and 9, a novel approach will be presented which does not call for making any arbitrary assumptions regarding the nature and structure of the parameters . Moreover, it also allows us to calculate time variation of the parameters over discrete intervals of time.

2.5 Some Special Grinding Phenomena

Based on the analysis of experimental data, several special grinding phenomena have been reported from time to time . In the previous sections , mention has already

been made of some of these - (1) difference - similar breakage parameters, (ii) Rosin - Rammler Kinetics, (iii) Alyavdin Kinetics and (iv) a power law for size variation of the rate parameters . We present below a brief review on two of the more important topics - (i) first order disappearance kinetics and (ii) zero order production of fines , which was not discussed earlier .

In a recent review, Austin |28| says that ''the first order disappearance of material in a discrete size interval is a hypothesis which has been proved as a reasonable first approximation ''. It is true that a large number of studies can be cited in support of this hypothesis | 25, 48, 50, 81, 82, 83 | , but at the same time evidences can also be produced which go against it | 41, 42, 84 | . Austin and Luckie | 72 | have shown that this hypothesis is valid for a suitably small size interval only . Harris | 45 | has also pointed out that the ln $M_1(t)$ vs. t plots are straight lines for some systems and not in case of others . Both Harris and Austin have attributed these deviations from the first order kinetics to many factors . Some of these are - (i) width of discrete size intervals | 45, 72 |, (ii) ratio of the particle size to the mill diameter and ball diameter |- 45, 46, 80 |, (iii) interactions between the

constant rate ,| 31, 42, 58, 83, 85, 86 |, i.e.

$$\frac{dF(x,t)}{dt} = \varphi(x) \qquad (2.124)$$

where $\varphi(x)$ is the formation rate constant for the material finer than size x . This phenomenon was first noted by Fahrenwald | 85 | . For a batch ball milling of quartz , Arbitier and Bhrany | 83 | found that

$$\varphi(x) = k_o \left(\frac{x}{x'}\right)^{\alpha} \qquad (2.125)$$

where x' and α are size modulus and distribution modulus of that portion of the product size distribution which is described by the Gaudin - Schumann distribution, and k_o is a constant . Fuerstenau and Somasundaran | 86 | , and Herbst and Fuerstenau | 58 | found that for limestone and dolomite also, the relation in Eq. (2.125) is valid.

It has been pointed out that for Eq. (2.124) to hold , the grinding of the fine particles themselves must be negligible | 42, 58, 86 | . Mika et al. | 42 | have rightly argued that an additional requirement is that the fine size fractions do not influence the breakage of coarser material in the appropriate time interval.

Harri s | 31 | showed that for small values of x and t , Eqs. (2.124) and (2.125) can be drived from Eq. (2.54) if g(x) is assumed to be of the form

$$g(x) = \exp(-x/x')^{\alpha} \qquad (2.126)$$

Mika et al. | 42 | and Herbst and Fuerstenau | 58 | proved that the condition given in Eq. (2.109) is sufficient to ensure the relationship given in Eq. (2.116), which is the size discretized version of Eq. (2.124) . It will be seen that the condition in Eq. (2.126) necessarily implies that Rosin-Rammler type kinetics prevails . It will be shown in Chapter 7 that the condition in Eq. (2.109) also necessarily implies Rosin-Rammler type kinetics . It then follows that the systems exhibiting the phenomenon of zero-order production of fines must conform to Rosin-Rammler type of kinetics, which surprisingly is not the case in both studies | 42, 58 | of Mika et al. | 42 | and Herbst and Fuerstenau | 58 | .

In a later publication | 45 | , Harris pointed out that the conventional method of plotting F(x,t) vs. t on an arithmetic grind is insensitive, and a close inspection of the grinding data reveals that it exhibits a systematic departure from Eq. (2.124) . He suggested that the following equation seems to be more appropriate for describing the data

$$F(x,t) \propto x^{\alpha} t^{(1+a)} \qquad (2.127)$$

A formal derivation of Eq. (2.127) from Eq. (2.65) has been given by Harris in an earlier publication | 38 | . Employing data from various sources, Harris has shown that the constant

'a' normally lies in the range - 0.3 < a < 0.3 .

It is quite evident from the above discussion that the two hypotheses of first order disappearance of a discrete size fraction and zero-order production of fines, are valid only in a very approximate sense . Harris | 45 | is right in proposing that these should be discarded as the general hypotheses . Further discussion on these two phenomena shall be given in Chapter 7. It may be mentioned briefly that the observed deviations are only natural consequences of violation of the Rosin-Rammler type of grinding kinetics .

2.6 Empirical Correlations for Variation in Rate Parameters With Various Operating Variables

A knowledge of the quantitative correlations for variation in the model parameters is quite important from the point of view of simulation studies and also for establishing the optimal working conditions. To date the effects of particle load | 25, 42, 82, 87 - 91 | , ball load | 91-93 |, mill speed | 92 | , ball diameter | 82, 94 - 98 | , grinding media shape | 99 | and grinding media density | 93 | have been investigated . A brief account of these findings is presented below .

Bowdish | 82 | found that the rate parameter S decreases as the hold-up of the particulate mass W is increased . For W corresponding to 100 per-cent filling of the interestitial volume of the static ball charge, the value of the absolute grinding rate SW was found to be maximum . This result is in agreement with the findings of Kapur and Agrawal | 88 | and Mika et al. | 42 | . Nijman | 89, 90 |, Mika et al. |42 | Kelsall et al. | 56, 87 | and Austin et al. | 91 | have shown that the rate parameter was inversely proportional to the particulate mass W when the particle loadings exceeded 100 per-cent filling of the interstial volume of the static ball charge . In the unsaturated regime (that is, less than 100 per-cent filling), the rate parameter was found to be approximately inversely proportional to the square root of the particulate mass | 42 | . However, a single correlation, valid for all size intervals andfor both starved and saturated mills, has not been presented so far . In particular, it has not been explicitly demonstrated whether the particulate mass affects the rate parameters for all size intervals of interest to the same extent . That this is indeed the case we shall show in Chapter 10 .

Herbst and Fuerstenau | 92 | found the rate parameter to be independent of the ball load, when the latter was less than 40 per-cent of the load required to fill the mill

completely and the particle load corresponded to 100 per-cent filling of the interstices of the ball mass at rest . For higher ball loads the value of the rate parameter was found to decrease, which was attributed to ball-ball interactions . The results of Herbst and Fuerstenau | 92 | as well as Austin et al. | 91 | show that the absolute grinding rate SW increased approximately linearly with ball load for small loads and attained a maximum at a load corresponding to 50 per-cent of that required for complete filling of the mill . Increasing the ball load beyond this percentage did not result in any change in the absolute grinding rate . The broad trend of results obtained by Kelsall et al. | 93 | is also the same .

Herbst and Fuerstenau | 92 | have reported results on variation of rate parameter with mill speed at constant particle and ball loads . Rate parameter was found to increase approximately linearly with increasing speed of rotation, attaining a maximum at about 80 per-cent of the critical speed, and then decreasing sharply as the mill speed approached 100 per-cent of the critical speed. According to Rose | 100 | , at slow speeds of rotation the dynamic configuration of the ball mass

changes little with increasing speed and therefore, the rate parameter on a per revoluation basis is expected to remain essentially constant for these conditions as has been observed by Herbst and Fuerstenau . Based on a mathem -atical theory of ball motion due to Davis | 101 | , for the experimental conditions employed by these authors, the best operating speed is calculated as 80 percent of the critical speed. This also agrees well with the experimental findings of Herbst and Fuerstenau .

Bowdish | 82 | studied the effect of ball size, keeping total weight of balls constant, on the rate parameters for different sizes of limestone feed . He found that the rate parameter were directly proportional to the surface area of the balls . However, later studies | 89, 90, 94,-98 | have confirmed that for a given mill, material, particle and ball loads, there is an optimal ball size for the maximum rate of grinding of a given type and size of particles. That is, for a given particle size, the rate parameter first increases with increase in ball diameter, reaches a sharp maximum and then decreases . Moreover, the curves relating rate parameter to ball diameter have been found to be similar in shape for all particle sizes, and both the optimal ball size

and the corresponding value of the rate parameter have been found to increase with increasing particle size. The portion of the curve in which the rate parameter decreases with increase in ball diameter has been found to be of the form $S_i \propto B^{-1}$ | 96 | and $S_i \propto B^{-2}$ | 94 |, where B is the ball diameter,. However, no general expressions are available relating the rate parameter as function of ball and particle size.

In addition to the studies summarized above, Kelsall et al. have reported results on the effect of grinding media shape | 99 | and grinding media density | 93 | also. However, the data presented is not sufficient for deriving quantitative correlations.

It should be mentioned that of all the variables discussed above only mill speed and grinding media shape have been reported to have any significant effect on the breakage parameters | 92, 99 | . Again, no quantitative correlations are available for describing the variation of breakage parameters with these two variables.

CHAPTER 3

OUTLINE OF WORK

The present work can be conveniently described under seven subtitles as follows :

(I) A Pseudo-Similarity Solution to the Integro-Differential Equation of Batch Grinding

With a view to obtaining a convenient closed form analytical solution to the fundamental integro-differential equation of batch grinding for more general forms of the selection and breakage functions than those used hitherto by a number of investigatiors | 14, 24, 30, 31 | , a new solution has been obtained using Kapur's similarity solution and standard mathematical transformation techniques . The resulting size spectra exhibit a non-self-preserving nature, which has been illustrated by simulation studies . The solution is the most general close form solution available at present .

(II) Exact Mathematical Definitions of the Parameters in the Discrete Size Grinding Models

As pointed out earlier in Chapter 2, most of the investigators have either formulated the discrete size models without any reference to the more fundamental

size continuous model and the two basic functions $S(x)$ and $\bar{B}(v,x)$, or they have failed to obtain the correct relationships for the discrete model parameters in terms of the two basic functions . This has led to considerable confusion and misinterpretation of many experimental data . With a view to providing a proper framework for formulation of the discrete size models, the exact mathematical definitions of the discrete size parameters have been derived from the fundamental integro-differential equation of batch grinding using a formal mathematical procedure . It has been shown that all the parameters are interrelated in a unique manner through the functions $S(x)$, $\bar{B}(v,x)$ and $M(x,t)$, and hence, cannot be assigned any arbitrary values, independent of each other .

A new function $L(v,x) = S(v)\,\bar{B}(v,x)$, which plays a central role in the analysis of discrete size models, has been identified and its important properties have been delineated.

(III) Realizations of Various Specialized Discrete Size Grinding Models

It is possible to simplify the expressions for size discretized parameters by specializing the functional forms of the functions $M(x,t)$ and $L(v,x)$. Three such specialized cases have been identified and the important

characteristics of the parameters of the resulting models have been delineated .

(IV) Analysis of Some Special Grinding Phenomena

The phenomena of first order disappearance kinetics ,zero-order production of fines, Alyavdin kinetics, difference similar breakage parameters, and variation of the rate parameters with particle size according to a power law, have been investigated in light of the structure of the three specialized discrete size models proposed. The pertinent discrete size models associated with these phenomena have been identified and delimited in this context.

(V) Identification of the Grinding System in Terms of a Mill Function

A novel mill function approach is presented for identification of a linear and time-invariant batch grinding system, in which size reduction occurs by impact fracture, chipping and abrasion. The approach is based on a size continuous and discrete time representation of the grinding system. The proposed identification scheme has been demonstrated for a batch ball mill - dolomite grinding data .

(VI) Estimation of Size Discretized Parameters

A '1,0' variable suppression technique has been devised for exact back calculation of all the size discretized parameters S_i and $B_{i,j}$ from a knowledge of the input-output

matrix $\underline{\underline{O}}$ in Eq. (2.103). Using the estimated mill functions for two sets of batch grinding data for dolomite, the $\underline{\underline{O}}$ matrices have been generated over several successive intervals of fixed grinding time and, for first time, the time variation of the model parameters due to their dependence on the size distribution of the particles within the individual discrete size intervals, has been successfully computed.

Based on mill-function a new set of parameters have been defined and named as 'reference parameters' . These are useful from the point of view of comparing different grinding systems.

(vii) Empirical correlations for Variation of Size - Discretized Rate Parameters With Particulate Mass and Ball Size

Systematic graphical procedures for determination of the empirical correlations for the variations in time independent rate parameters (Bass Model) with particulate nass and the size of the grinding media ball have been leveloped and successfully demonstrated for two actual grinding systems.

CHAPTER 4

A PSEUDO-SIMILARITY SOLUTION TO THE INTEGRO-DIFFERENTIAL EQUATION OF BATCH GRINDING

We have seen in Chapter 2.3.3 that it has not been possible to obtain a convenient closed form analytical solution to the fundamental integro-differential equation of batch grinding (Eq. 2.36) for general forms of the selection and breakage functions . On the other hand , the specialized mathematical forms of the selection and breakage functions used by Gaudin and Meloy |24| , Austin et al. |30| , Harris |31| and Kapur |14| seem to be much too restricted to be realistic for a given mill-material system. In this Chapter an attempt has been made to extract solutions to Eq.(2.36) for a more general class of the selection and breakage functions . The strategy involves a suitable mathematical transformation of particle size x into a pseudo-size $\bar{u}$, a similarity solution to the grinding equation in $\bar{u}$, followed by back transformation into x, resulting in a pseudo-similarity solution to the grinding equation in particle size .

4.1 Pseudo-Similarity Solution

Let the grinding functions be given by

$$S(x) = A\,[\,g(x)\,]^{\alpha} \qquad (4.1)$$

and

$$\bar{B}(v,x) = \left[\frac{g(x)}{g(v)} \right]^{\beta} \tag{4.2}$$

where g is any monotonically non-decreasing function in particle size such that $g(0) = 0$. Substitution of Eqs.(4.1) and (4.2) into Eq. (2.36) gives

$$\frac{\partial M(x,t)}{\partial t} = - A \left[g(x) \right]^{\alpha} M(x,t) + \int_0^{\infty} A\left[g(v)\right]^{\alpha} \frac{\partial}{\partial x} \left[\frac{g(x)}{g(v)} \right]^{\beta} M(v,t)\, dv \tag{4.3}$$

We seek a solution to this equation by a suitable one to one transformation of the variable x . Let us define a transformed variable , the pseudo-size $\bar{u}$, in the following manner

$$\bar{u} = g(x) \tag{4.4}$$

and

$$\bar{w} = g(v) \tag{4.5}$$

Hence , the distribution in pseudo-size $Y(\bar{u}, t)\, d\bar{u}$, defined as the mass fraction of particles with attribute in the range of $\bar{u}$ to $\bar{u} + d\bar{u}$, is related to $M(x,t)\, dx$ by

$$M(x,t) = Y(\bar{u}, t) \left| \frac{d\bar{u}}{dx} \right| \tag{4.6}$$

or

$$M(x,t) = Y\left[g(x), t \right] \left| \frac{dg(x)}{dx} \right| \tag{4.7}$$

Moreover

$$\frac{\partial}{\partial x} \left[\frac{g(x)}{g(v)} \right]^{\beta} = \frac{\partial}{\partial \bar{u}} \left(\frac{\bar{u}}{\bar{w}} \right)^{\beta} \left| \frac{dg(x)}{dx} \right| \tag{4.8}$$

Substitution of Eqs. (4.4) - (4.8) into Eq. (4.3) gives

$$\frac{\partial Y(\bar{u},t)}{\partial t} = - A\, \bar{u}^{\alpha}\, Y(\bar{u},t) + \int_{\bar{u}}^{\infty} A\, \bar{w}^{\alpha}\, \frac{\partial}{\partial u} \left(\frac{\bar{u}}{\bar{w}} \right)^{\beta} Y(\bar{w},t)\, d\bar{w} \tag{4.9}$$

which has a similarity solution |14|

$$Y(\bar{u},t) = \frac{C_0}{\nu_1(t)} \left[\frac{\bar{u}}{\nu_1(t)} \right]^{\beta-1} \exp \left[- \frac{1}{\alpha h} \left(\frac{\bar{u}}{\nu_1(t)} \right)^{\alpha} \right] \tag{4.10}$$

where

$$\nu_1(t) = \int_0^{\infty} \bar{u}\, Y(\bar{u},t)\, d\bar{u} \tag{4.11}$$

Substitution of Eqs. (4.4) and (4.6) gives

$$\nu_1(t) = \int_0^{\infty} g(x)\, M(x,t)\, dx \tag{4.12}$$

Using Eqs. (4.4) and (4.7) , we back transform the distribution in $\bar{u}$ into the distribution in particle size x and obtain

$$M(x,t) = \frac{C_0}{\nu_1(t)} \left[\frac{g(x)}{\nu_1(t)} \right]^{\beta-1} \exp \left[- \frac{1}{\alpha h} \left(\frac{g(x)}{\nu_1(t)} \right)^{\alpha} \right] \left| \frac{dg(x)}{dx} \right| \tag{4.13}$$

which is the desired solution to Eq. (4.3) . It is shown in the Appendix 2 that when $g(x) = x^d$, we obtain the original similarity solution of Kapur from Eq. (4.13) , thereby confirming the validity of our approach . It should be noted that while the distribution in $\bar{u}$, $Y(\bar{u},t)$, is self-preserving, the distribution $M(x,t)$ in scaled variables $x/\nu_1(t)$ or $g(x)/\nu_1(t)$

is no longer self-similar because of the dg(x)/dx term present in Eq. (4.13) . This term is independent of x only when g(x) is a linear function of x . Thus under a linear transformation in Eq. (4.4) , the size distribution in $g(x)/\mathcal{V}_1(t)$ retains its self-similar character .

4.2 Simulation Results

In order to illustrate the shape of the computed size distribution curves of M(x,t) , we have assumed the following two forms for the function g(x)

$$g_1(x) = 0.5 \left[x + 10^{-3} x^2 \right] \qquad (4.14)$$

Hence

$$\frac{dg_1(x)}{dx} = 0.5 + 10^{-3} x \qquad (4.15)$$

And

$$g_2(x) = 10^3 \left[\exp (10^{-3} x) - 1 \right] \qquad (4.16)$$

Hence

$$\frac{dg_2(x)}{dx} = \exp (10^{-3} x) \qquad (4.17)$$

where particle size x is expressed in microns . Cumulative size distributions were computed in $\bar{u}$ domain using Eq. (4.10) with A = 1, $\alpha = 0.5$, $\beta = 1$, and $\mathcal{V}_1(0) = 5657$, for four times corresponding to $\mathcal{V}_1 = \mathcal{V}_1(0)$, $\mathcal{V}_1(0)/4$, $\mathcal{V}_1(0)/16$, and $\mathcal{V}_1(0)/64$. Distributions in x domain were obtained by transforming $\bar{u}$ into x using Eqs. (4.4) , (4.14) and (4.16) . Figures 4.1 and 4.2 show these distributions at various times on log-log grid . The shapes of the distributions are quite

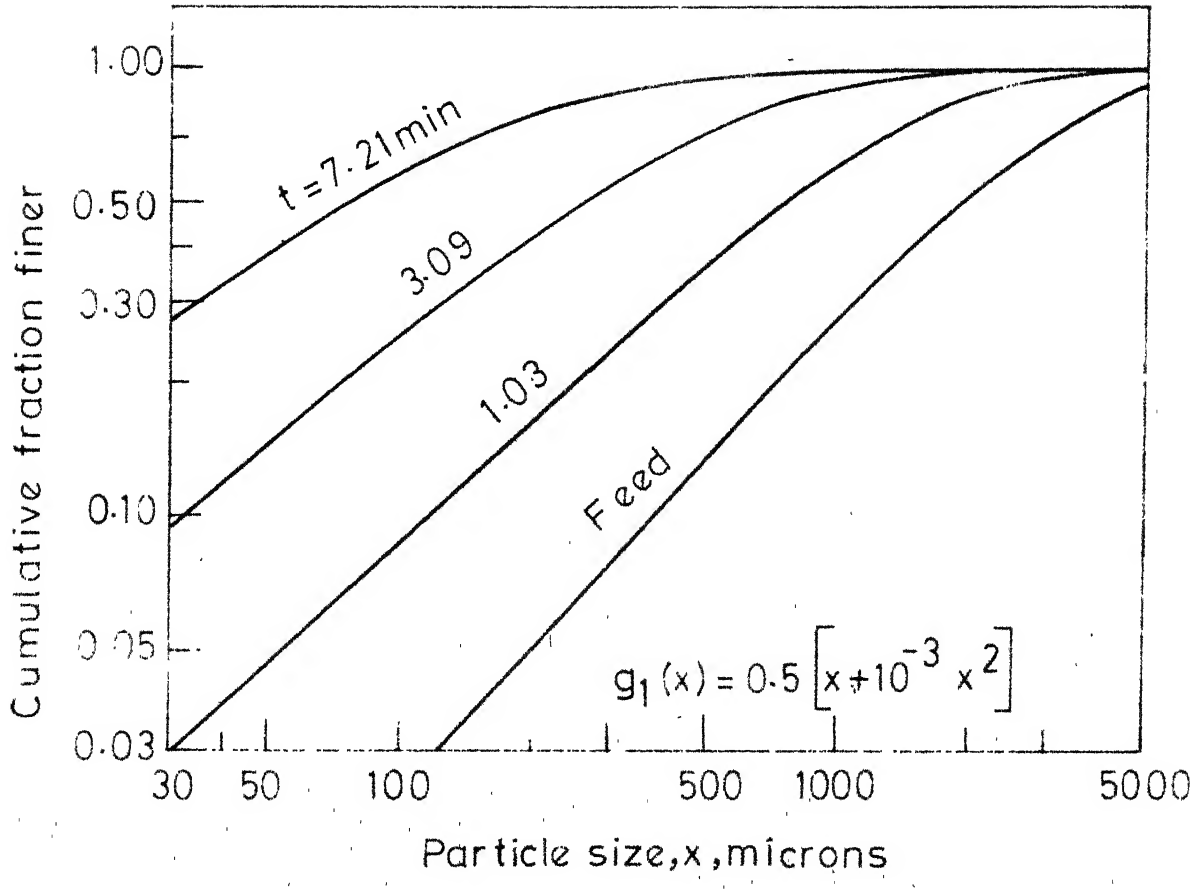

Fig. 4.1 Cumulative particle size distributions at various times for $g_1(x)$

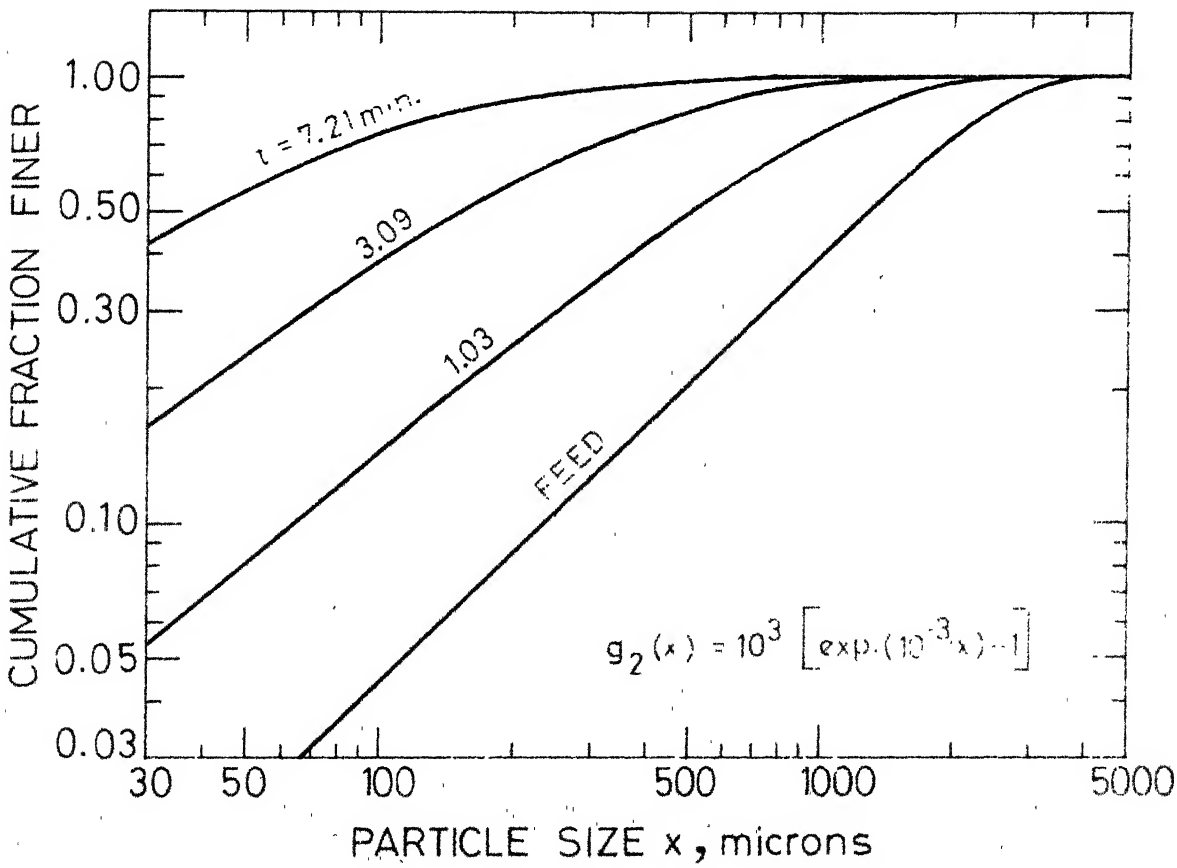

Fig. 4.2 Cumulative particle size distributions at various times for $g_2(x)$.

similar to those based on numerous experimental data reported in the literature .

In order to illustrate the nature of deviation from the self-preserving behaviour , we recall |14| that the shape of the similarity distribution in $\bar{u}$ domain is time invariant when the cumulative distribution function is plotted as a function of $\bar{u}/\bar{u}_{0.5}$, where $\bar{u}_{0.5}$ is the median size . Figures (4.3) and (4.4) for transformations g_1 and g_2 , respectively, show that the curves in particle size x do not collapse and are significantly non-similar . The dotted curves are the self - preserving forms in pseudo-size $\bar{u}$ domain . The nature and the extent of deviation from a self-similar shape depends on the type of transformation employed . With increasing grinding time more and more material concentrates in the finer size range . Inspection of Eqs. (4.14) and (4.16) shows that as x approaches zero , the transformations tend to become linear . Therefore , the distributions in x should tend to the self-preserving form with increasing grinding time . This is readily confirmed from Figs. (4.3) and (4.4) , where it can be seen that pseudo-similarity curves eventually merge with the self-preserving curves in $\bar{u}$ domain .

.3 Discussion

The frequency function of the comminuted particles in

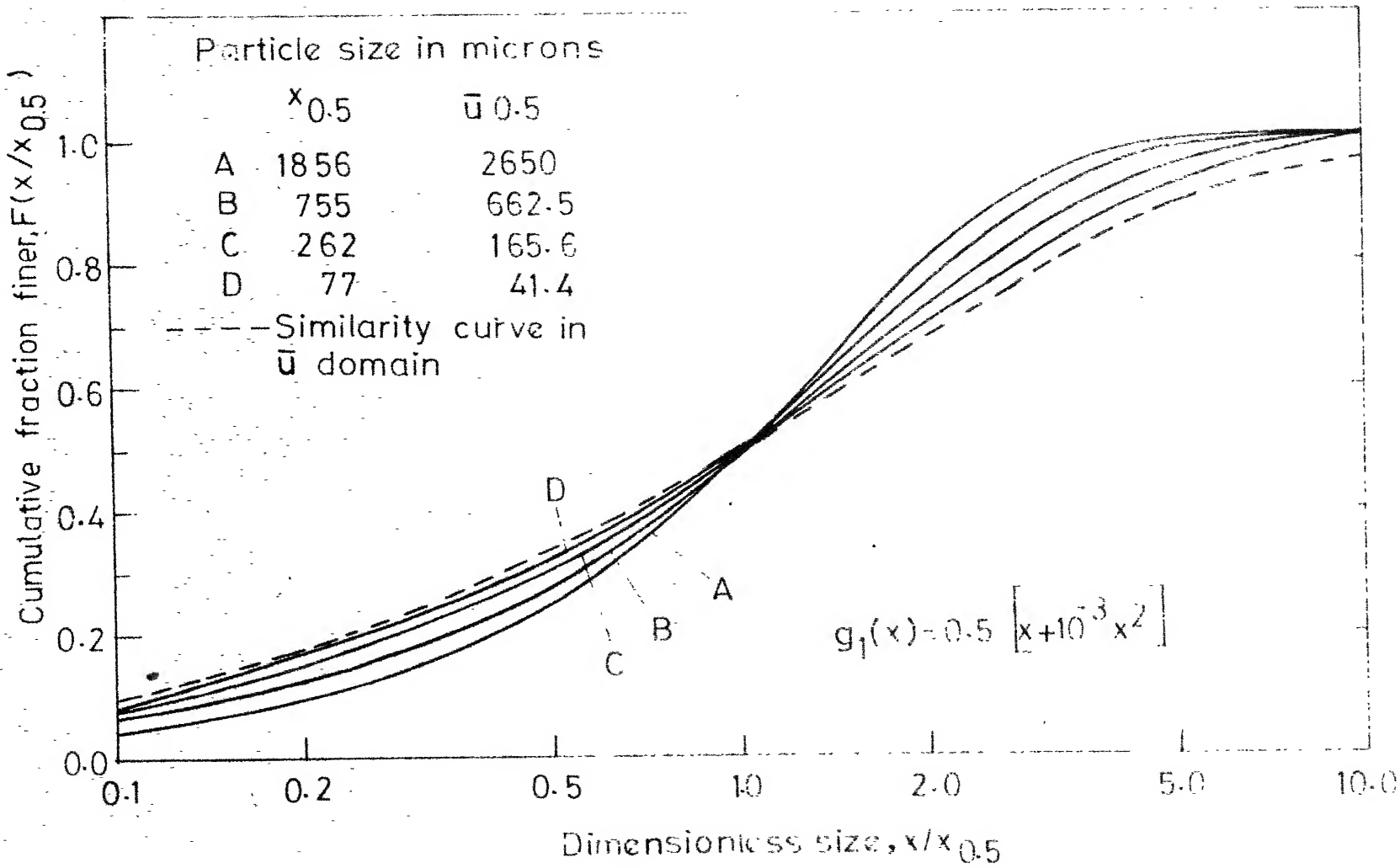

Fig 4.3 The nature of deviations from self preserving behaviour for $A=1$, $\alpha=0.5$, $\beta=1$ and $g_1(x)$.

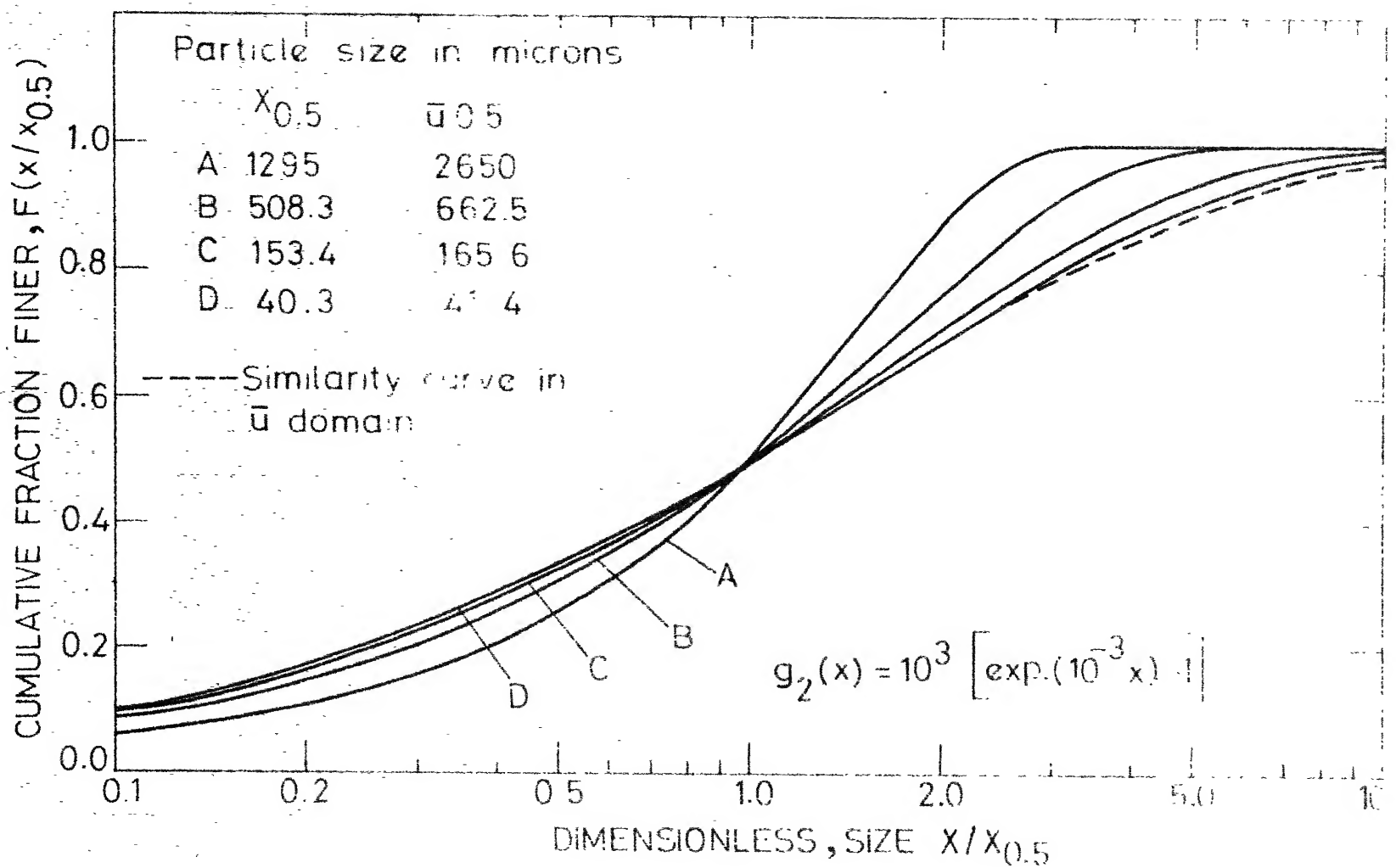

Fig. 4.4 The nature of deviations from self preserving behaviour for $A = 1$, $\alpha = 0.5$, $\beta = 1$ and $g_2(x)$.

Eq. (4.13) is the most general analytical expression ever derived from a formal kinetic grinding model. It is much more elastic and versatile than the Rosin-Rammler and the generalized gamma distributions . The equation describes the complete trajectory of the particle spectra in time , apart from a small initial period . As such its importance , specially for simulation studies , is self evident . The combined transformation - similarity solution method presented here is of great utility ; it can be employed for obtaining new solutions to a variety of particle population problems [103, 104] in the way similar to the one described above . .

CHAPTER 5

EXACT MATHEMATICAL DEFINITIONS OF THE PARAMETERS IN THE DISCRETE SIZE GRINDING MODELS

In view of the discussion in Chapter 2 on the current status of the literature on discrete size grinding models , it is quite evident that the establishment of the correct mathematical definitions of the discrete size parameters in terms of the two basic size continuous functions, $S(x)$ and $B(v,x)$, is a task of some importance . In this chapter therefore , the required expressions for the parameters of the two size discretized models have been obtained using a formal mathematical procedure , which consists of first an appropriate size discretization of the fundamental integro-differential equation of batch grinding and then a term by term comparison with the two master kinetic equations for discrete size models .

5.1 Derivation of the Exact Mathematical Definitions of the Parameters for Model 'A'

Substitution of Eq. (2.34) in the following identity

$$\frac{dM_i(t)}{dt} = \frac{d\left[F_{i-1}(t) - F_i(t) \right]}{dt} \tag{5.1}$$

gives

$$\frac{dM_i(t)}{dt} = \int_{x_{i-1}}^{x_0} S(v)\, \bar{B}\,(v,\, x_{i-1})\, M(v,t)\, dv$$

$$- \int_{x_i}^{x_0} S(v)\, \bar{B}\,(v\,,\, x_i)\, M\,(v,t)\, dv \tag{5.2}$$

Break up of the integrals over discrete size intervals and rearrangement of the resulting terms gives

$$\frac{dM_i(t)}{dt} = -\int_{x_i}^{x_{i-1}} S(v)\ \bar{B}\,(v,x_i)\, M(v,t)\, dv$$

$$+ \sum_{j=1}^{i-1} \int_{x_j}^{x_{j-1}} S(v)\left[\bar{B}(v,x_{i-1}) - \bar{B}(v,x_i) \right] M(v,t)\, dv \tag{5.3}$$

Term by term comparison of Eqs.(2.79) and (5.3) shows that

$$S_i(t)\, M_i(t) = \int_{x_i}^{x_{i-1}} S(v)\, \bar{B}(v,x_i)\, M(v,t)\, dv \tag{5.4}$$

Hence , the instantaneous rate parameter is

$$S_i(t) = \frac{1}{M_i(t)} \int_{x_i}^{x_{i-1}} S(v)\, \bar{B}(v,x_i)\, M(v,t)\, dv \tag{5.5}$$

We note that the mass fraction of solids in size interval i is

$$M_i(t) = \int_{x_i}^{x_{i-1}} M(x,t)\, dx\,, \quad i = 1\,,\ 2\,,\ \ldots \qquad (5.6)$$

Hence , an alternative equivalent expression for the rate parameter in terms of the size continuous functions can be wirtten by combining Eqs. (5.5) and (5.6)

$$S_i(t) = \frac{\int_{x_i}^{x_{i-1}} S(v)\ \bar{B}(v,x_i)\ M(v,t)\ dv}{\int_{x_i}^{x_{i-1}} M(v,t)\ dv} \qquad (5.7)$$

In same manner , the expression for the instantaneous breakage parameter is

$$B_{i,j}(t) = \frac{\int_{x_j}^{x_{j-1}} S(v)\ [\ \bar{B}(v,x_{i-1}) - \bar{B}(v,x_i)\ ]\ M(v,t)\ dv}{\int_{x_j}^{x_{j-1}} S(v)\ \bar{B}(v,x_j)\ M(v,t)\ dv}$$

$$, \ j \neq i \qquad (5.8)$$

Substitution of Eq. (5.8) in Eq. (2.82) gives the following expression for the instantaneous cumulative breakage parameter

$$\bar{B}_{i,j}(t) = \frac{\int_{x_j}^{x_{j-1}} S(v)\ \bar{B}(v,x_i)\ M(v,t)\ dv}{\int_{x_j}^{x_{j-1}} S(v)\ \bar{B}(v,x_j)\ M(v,t)\ dv} \qquad (5.9)$$

The relationships in Eqs. (5.7) , (5.8) and (5.9) are of great importance . Much of the controversy and confusion - frequently leading to erroneous conclusions - prevailing in the recent grinding literature may be attributed to our failure to recognize these relationships and derive the correct conclusions therefrom . It should be noted that Eqs. (5.7) and (5.8) clearly establish that - (i) the parameters S_i and $B_{i,j}$ are not independent of each other as both are determined by the three functions $S(x)$, $\bar{B}(v,x)$ and $M(x,t)$ jointly , (ii) there is no one - to - one physical correspondence between $S(x)$ and S_i or between $B(v,x)$ and $B_{i,j}$, and (iii) the parameters are functions of the instantaneous particle size distribution and , hence , vary with time . In view of the last abservation , it becomes necessary to rewrite the Eq. (2.79) as follows

$$\frac{dM_i(t)}{dt} = - S_i(t)\, M_i(t) + \sum_{j=1}^{i-1} S_j(t)\, B_{i,j}(t)\, M_j(t) \quad , \; i = 2 , 3, \; .... \quad (5.10)$$

It is interesting to note that even though the grinding system has been assumed to be linear and time - invariant , the size discretized representation of the kinetic model is that of a time varying system . This apparent anamoly can , however , be easily resolved . It should be recalled

that the model parameters refer to a size range of particles and not a single particle size . Also, the contribution to the value of a model parameter of each particle size within a discrete size interval , is proportional to the relative mass density of particle population at that size . Since, the latter varies with grinding time , it is easily seen that the time variability of the model parameters is inherent in the discretization procedure itself and , therefore , cannot be avoided .

5.2 Derivation of the Exact Mathematical Definitions of the Parameters for Model 'B'

Let us discretize the integral term on the right hand side in Eq. (2.36) as follows

$$\frac{\partial M(x,t)}{\partial t} = - S(x)\, M(x,t) + \int_{x}^{x_{i-1}} S(v)\, \frac{\partial \bar{B}(v,x)}{\partial x}\, M(v,t)\, dv + \sum_{j=i-1}^{1} \int_{x_j}^{x_{j-1}} S(v)\, \frac{\partial \bar{B}(v,x)}{\partial x}\, M(v,t)\, dv \qquad (5.11)$$

Now, we integrate Eq. (5.11) on both sides over a size interval $x_i \leq x \leq x_{i-1}$, hence

$$\frac{\partial \int_{x_i}^{x_{i-1}} M(x,t)\, dx}{\partial t} = - \int_{x_i}^{x_{i-1}} S(x)\, M(x,t)\, dx + \ldots$$

$$\ldots + \int_{x_i}^{x_{i-1}} \int_{x}^{x_{i-1}} S(v) \frac{\partial \bar{B}(v,x)}{\partial x} M(v,t)\, dv\, dx$$

$$+ \sum_{j=i-1}^{1} \int_{x_i}^{x_{i-1}} \int_{x_j}^{x_{j-1}} S(v) \frac{\partial \bar{B}(v,x)}{\partial x} M(v,t)\, dv\, dx \qquad (5.12)$$

Now, using Eq. (5.6) and carrying out a term by term comparison with the model equation Eq. (2.86), we obtain

$$s_i(t) = \frac{\int_{x_i}^{x_{i-1}} S(x)\, M(x,t)\, dx}{\int_{x_i}^{x_{i-1}} M(x,t)\, dx} \qquad (5.13)$$

$$b_{i,i}(t) = \frac{\int_{x_i}^{x_{i-1}} \int_{x}^{x_{i-1}} S(v) \frac{\partial \bar{B}(v,x)}{\partial x} M(v,t)\, dv\, dx}{\int_{x_i}^{x_{i-1}} S(v)\, M(v,t)\, dv} \qquad (5.14)$$

$$b_{i,j}(t) = \frac{\int_{x_i}^{x_{i-1}} \int_{x_j}^{x_{j-1}} S(v) \frac{\partial \bar{B}(v,x)}{\partial x} M(v,t)\, dv\, dx}{\int_{x_j}^{x_{j-1}} S(v)\, M(v,t)\, dv} \quad , i \neq j \qquad (5.15a)$$

which can be simplified as

$$b_{i,j}(t) = \frac{\int_{x_j}^{x_{j-1}} S(v) \left[\bar{B}(v, x_{i-1}) - \bar{B}(v, x_i) \right] M(v,t)\, dv}{\int_{x_j}^{x_{j-1}} S(v)\, M(v,t)\, dv} \quad , \; i \neq j \qquad (5.15)$$

We recall that by definition

$$b_{i,i}(t) = 1 - \sum_{j=i+1}^{\infty} b_{j,i}(t) \qquad (5.16)$$

Thus , by substituting Eq. (5.15) in Eq. (5.16) , a relatively simple expression is obtained for the inner breakage parameter

$$b_{i,i}(t) = 1 - \frac{\int_{x_i}^{x_{i-1}} S(v)\, \bar{B}(v, x_i)\, M(v,t)\, dv}{\int_{x_i}^{x_{i-1}} S(v)\, M(v,t)\, dv} \qquad (5.17)$$

It can be confirmed by carrying out some mathematical manipulations that Eqs. (5.14) and (5.17) are identical .

A comparison of the expressions derived above with those mentioned by Whiten |54| will reveal that his equations are grossly incorrect . Similarly, by substituting

the relationships used by Olsen |53| for $S(v)$ and $\bar{B}(v,x)$ and evaluating the integrals in the expressions for s_i, $b_{i,i}$ and $b_{i,j}$, it can be shown that these results do not match with the corresponding expressions derived by Olsen|53| and used by Olsen and Krogh |71| . Consequently, the analysis of the grinding operation as presented by these authors is suspect and needs a revision .

It may also be emphasized that both the rate and breakage parameters S_i and $B_{i,j}$, and, the selection and distribution parameters s_i and $b_{i,j}$ are time dependent i.e., the latter set of parameters are also functions of the particle size distribution within the individual size intervals .

Henceforth in this work we shall restrict ourselves mainly to the more popular model i.e., the model 'A' .

5.3 The Selekage Function

We note that in the expressions derived above for model 'A', the selection and breakage functions occur in association only . The product term $S(v)\ \bar{B}(v,x)$, which will appear repeatedly in the discrete size analysis, deserves a name of its own . We shall call it as the 'selekage function $L(v,x)$', that is

$$L(v,x) = S(v)\ \bar{B}(v,x) \qquad (5.18)$$

We mention in passing that the differential form of the selekage function

$$\frac{\partial L(v,x)}{\partial x} = S(v)\ B(v,x) \qquad (5.19)$$

$$= G(v,x)$$

first appeared in the integro - differential equation of batch grinding formulated by Bass |22| .

We now present some of the more important properties of the selekage function . Conservation of mass requires that the breakage distribution function be a normalized one , hence

$$\bar{B}(v,v) = 1 \qquad (5.20)$$

From definition of the selekage function in Eq. (5.18) we have .

$$S(v) = L(v,v) \qquad (5.21)$$

and

$$\bar{B}(v,x) = \frac{L(v,x)}{L(v,v)} \qquad (5.22)$$

Moreover ,

$$B(v,x) = \frac{1}{L(v,v)} \quad \frac{\partial L(v,x)}{\partial x} \qquad (5.23)$$

In principle then the knowledge of the selekage function is equivalent to the knowledge of the selection and breakage functions .

The cumulative breakage function $\bar{B}(v,x)$ is a monotonically non-decreasing function of x and if S(v) can assume only positive definite values , it follows that the selekage function is also restricted to positive definite values ,only . Moreover

$$\bar{B}(v,0) = 0 \tag{5.24}$$

therefore

$$L(v,0) = 0 \tag{5.25}$$

5.4 The Master Equations in Terms of the Selekage Function

For the reasons mentioned above , in the analysis of the discrete size grinding models , it is preferable to write the mathematical expressions in terms of the selekage function . For ready reference , it is therefore convenient to rewrite some of the important equations replacing the product term $S\bar{B}$ by L . Consequently , the forms of Eqs.(5.3), (5.7) , (5.8) , (5.9) and (5.17) are , respectively .

$$\frac{dM_i(t)}{dt} = \int_{x_i}^{x_{i-1}} L(v, x_i)\, M(v,t)\, dv + \sum_{j=1}^{i-1} \int_{x_j}^{x_{j-1}} [L(v,x_{i-1}) - L(v,x_i)]\, M(v,t)\, dv \tag{5.26}$$

$$S_i(t) = \frac{\int_{x_i}^{x_{i-1}} L(v,x_i)\, M(v,t)\, dv}{\int_{x_i}^{x_{i-1}} M(v,t)\, dv} \qquad (5.27)$$

$$B_{i,j}(t) = \frac{\int_{x_j}^{x_{j-1}} [L(v,x_{i-1}) - L(v,x_i)]\, M(v,t)\, dv}{\int_{x_j}^{x_{j-1}} L(v,x_j)\, M(v,t)\, dv} \quad , i \neq j \qquad (5.28)$$

$$\bar{B}_{i,j}(t) = \frac{\int_{x_j}^{x_{j-1}} L(v,x_i)\, M(v,t)\, dv}{\int_{x_j}^{x_{j-1}} L(v,x_j)\, M(v,t)\, dv} \qquad (5.29)$$

$$b_{i,i}(t) = 1 - \frac{\int_{x_i}^{x_{i-1}} L(v,x_i)\, M(v,t)\, dv}{\int_{x_i}^{x_{i-1}} S(v)\, M(v,t)\, dv} \qquad (5.30)$$

CHAPTER 6

REALIZATIONS OF VARIOUS SPECIALIZED DISCRETE SIZE MODELS

For an inherently linear and time-invariant system the general most discrete size model is given by Eqs.(5.10), (5.27) and (5.28) in which both sets of grinding parameters are non-unique because of their dependence on the instantaneous particle size distribution . It is recognised that the grinding equations for the General model present formidable mathematical complexities . Therefore , there exists a geniune need to explore the possibilities of simplification in the grinding model . In this chapter three such cases have been identified and the characteristics of the resulting specialized discrete size models have been delineated .

6.1 The Bass Model

The Bass model, which is given by Eqs. (2.40) , (2.72) and (2.73) , is characterized with time-invariant parameters $v_{i,j}$. An alternate derivation of this model is possible using somewhat different arguments . Let us assume that we are justified in approximating the particle size distribution within the discrete size intervals by a uniform

distribution . This implies

$$M(x,t) = d_i(t) \quad , \quad x_i \leq x \leq x_{i-1} \tag{6.1}$$

and

$$d_i(t) = \frac{M_i(t)}{x_{i-1} - x_i} \tag{6.2}$$

Substituting these equations into Eqs. (5.27) and (5.28) , the following results are obtained

$$S_i = \frac{\int_{x_i}^{x_{i-1}} L(v,x_i)\, dv}{x_{i-1} - x_i} \tag{6.3}$$

$$B_{i,j} = \frac{\int_{x_j}^{x_{j-1}} \left[L(v,x_{i-1}) - L(v,x_i) \right] dv}{\int_{x_j}^{x_{j-1}} L(v,x_j)\, dv} \tag{6.4}$$

Hence ,

$$S_j\, B_{i,j} = \frac{\int_{x_j}^{x_{j-1}} \left[L(v,x_{i-1}) - L(v,x_i) \right] dv}{x_{i-1} - x_i} \tag{6.5}$$

From Eq. (5.19) it follows that

$$L(v,x_i) = \int_0^{x_i} S(v)\, B(v,x)\, dx \tag{6.6}$$

Substitution of Eq. (6.6) into Eq. (6.3) gives

$$S_i = \frac{\int_0^{x_i} \int_{x_i}^{x_{i-1}} S(v)\, B(v,x)\, dv\, dx}{x_{i-1} - x_i} \qquad (6.7a)$$

Discretizing the first integral Eq. (6.7a) can be rewritten as

$$S_i = \frac{\sum_{k > i} \int_{x_k}^{x_{k-1}} \int_{x_i}^{x_{i-1}} S(v)\, B(v,x)\, dv\, dx}{x_{i-1} - x_i} \qquad (6.7)$$

Following Bass |22| , let

$$v_{k,i} = \frac{\int_{x_k}^{x_{k-1}} \int_{x_i}^{x_{i-1}} S(v)\, B(v,x)\, dv\, dx}{x_{i-1} - x_i} \qquad (6.8)$$

We have

$$S_i = \sum_{k > i} v_{k,i} \qquad (6.9)$$

Similarly we can show that in Eq. (6.5)

$$S_j B_{i,j} = \frac{\int_{x_i}^{x_{i-1}} \int_{x_j}^{x_{j-1}} S(v)\, B(v,x)\, dv\, dx}{x_{j-1} - x_j} \qquad (6.10)$$

Hence

$$S_j \, B_{i,j} = v_{i,j} \tag{6.11}$$

Substitution of Eqs. (6.9) , (2.72) and (6.11) into the master equation Eq. (5.10) immediately leads to the size discretized time invariant parameter grinding model in Eq. (2.73) i.e., the Bass model .

Bass |22| had suggested that the inequality in Eq. (2.74) concurrently implies a vanishingly small inner breakage parameter $b_{i,i}$. In order to check this assertion we substitute Eq. (6.1) into Eq. (5.30)

$$b_{i,i} = 1 - \frac{\int_{x_j}^{x_{j-1}} L(v,x_j)\, dv}{\int_{x_j}^{x_{j-1}} S(v)\, dv} \tag{6.12}$$

and to obtain some numerical values of $b_{i,i}$, let us assign - simply for the sake of convenience in computation - the widely used forms for the selection and breakage functions as given in Eqs. (2.56) and (2.57) . Therefore , the selekage function is

$$L(v,x_j) = A\, v^{\alpha-\beta}\, x_j^{\beta} \tag{6.13}$$

Substitution of Eqs. (2.56) and (6.13) into Eq. (6.12) gives

$$b_{i,i} = 1 - \left[\frac{\alpha + 1}{\alpha - \beta + 1}\right]\left[\frac{\delta^{\alpha-\beta+1}-1}{\delta^{\alpha+1}-1}\right] \quad , \alpha - \beta + 1 \neq 0 \tag{6.14a}$$

$$= 1 - \left[\frac{\alpha + 1}{\delta^{\alpha+1} - 1}\right] \ln \delta \quad , \alpha - \beta + 1 = 0 \tag{6.14b}$$

which is independent of the size index . Table 1 gives values of the inner breakage parameter for various combinations of α and β for two sieve size ratios $\delta = \sqrt{2}$ and $\sqrt[4]{2}$ when we impose the Bass model on the grinding system .

It is clear that in general the inner breakage parameter is not insignificant and a substantial amount of material on breakage falls back into the original size interval even when the sieve size ratio is only $\sqrt[4]{2}$. The value of the inner breakage parameter is quite insensitive to the variation of the selection function with particle size but increases significantly with the slope of the breakage function , which is intutively reasonable . It can be concluded then that the sieve size ratio δ as small as $\sqrt[4]{2}$ is not adequate to satisfy the Bass condition .

However , as shown above , in order to generate the Bass model it is not necessary to stipulate that $b_{i,i} = 0$. The sufficient condition is that the particles are uniformly

distributed within a size interval . It is true that particles cannot remain uniformly distributed within the size intervals throughout the course of grinding , however , as a matter of practical necessity , we may arbitrarily impose a piecewise uniform distribution . This approximation becomes increasingly valid as the size interval width decreases . Also, it is physically more realistic because our approximation does not ignore the fact that the daughter particles indeed fall back into the original size interval .

As pointed out earlier in Chapter 2 the Bass model in Eq. (2.73) can also be represented by Eq. (2.79) , which is the most widely used discrete size grinding equation in the literature |42, 49, 50, 55 - 67| . However , two important differences should be noted in respect of the model formulation . First , unlike in the works of Mika et al. |42| and Herbst and Fuerstenau |58| , the derivation of Eq. (2.79) does not require the assumption of $b_{i,i}=0$. Second , parameters S_i and $B_{i,j}$, as given in Eqs. (6.3) and (6.4) , are not uncorrelated or independent of each other as regarded by most of the workers |42, 49 - 52, 55, 58, 61, 62, 73| . Both sets of parameters are functions of the selekage function and thus, are interrelated in a unique manner . It also follows that these parameters cannot be

equated with mutually independent phenomena of selection for breakage and breakage distribution function as originally proposed by Epstein |16| .

6.2 The RSF Model

A inspection of Eqs. (5.27) and (5.28) shows that it is not essential to assume a uniform distribution within a size interval , as done in the previous case , in order to generate a size discretized model with time invariant parameters ,. In some cases if it should turn out that the selekage function $L(v,x)$ is independent of variable v, that is

$$L(v,x) = A\, L_2\,(x) \tag{6.15}$$

where L_2 is some function of x only , then it can be easily shown that the form of this reduced selekage function is necessarily as given below

$$L(v,x) = A\, L_2\,(v)\ \frac{L_2(x)}{L_2(v)} \tag{6.16}$$

where now $A > 0$ is a constant .

Thus from Eqs. (5.21), (5.22) and (6.16)

$$S(v) = A\, L_2(v) \tag{6.17}$$

and

$$\bar{B}(v,x) = \frac{L_2(x)}{L_2(v)} \tag{6.18}$$

Since $\bar{B}(v,x)$ is a monotonically non-decreasing function of x, it follows that $L_2(x)$ and, hence, $S(x)$ is also a monotonically non-decreasing function of x. Again, combining Eq. (5.24) with Eq. (6.18) gives

$$L_2(0) = 0 \tag{6.19}$$

therefore

$$S(0) = 0 \tag{6.20}$$

Some of the results mentioned above were derived earlier by Austin et al. |30| .

Substitution of the reduced selekage function (RSF) into Eqs. (5.27) and (5.28) gives

$$S_i = A\ L_2(x_i) \tag{6.21}$$

and

$$B_{i,j} = \frac{L_2(x_{i-1}) - L_2(x_i)}{L_2(x_j)} \tag{6.22}$$

Substitution of these time-invariant parameters into Eq. (5.10), the master equation, leads to the size discretized RSF model as follows

$$\frac{dM_i(t)}{dt} = -A\,L_2(x_i)\,M_i(t) + \left[\,L_2(x_{i-1}) - L_2(x_i)\,\right] \sum_{j=1}^{i-1} M_j \tag{6.23}$$

The inner breakage parameter for the RSF model is

$$b_{i,i}(t) = 1 - \frac{A\, L_2(x_i) \int_{x_i}^{x_{i-1}} M(v,t)\, dv}{\int_{x_i}^{x_{i-1}} S(v)\, M(v,t)\, dv} \qquad (6.24)$$

which is not time - invariant . It will be seen that if $\alpha = \beta$, Eqs. (2.56) and (2.57) reduce to RSF model with $L_2(x) = x^{\alpha}$, and $S(x) = A\, x^{\alpha}$. Now , Kapur's similarity solution |14| can be used to compute the time variation of $b_{i,i}$. Substitution of Eqs. (2.59) into Eq. (6.24) gives an explicit expression for $b_{i,i}(t)$ as follows

$$b_{i,i}(t) = 1 - \frac{x_i^{\alpha} \int_{x_i}^{x_{i-1}} v^{\alpha-1} \exp\left[-\frac{1}{\alpha h}\left(\frac{v}{\mu_1(t)}\right)^{\alpha}\right] dv}{\int_{x_i}^{x_{i-1}} v^{2\alpha-1} \exp\left[-\frac{1}{\alpha h}\left(\frac{v}{\mu_1(t)}\right)^{\alpha}\right] dv} \qquad (6.25)$$

For simulation x_1 was assigned a value 1 and A, α, δ, and $\mu_1(0)$ were chosen 1, 1 , $\sqrt{2}$ and $\sqrt{2}/4$, respectively . Units of time used were minutes . For illustration of the results , x_1 has been denoted by 10 mesh sieve size . It is evident that in theory the similarity solution covers the

range $0 < x < \infty$, therefore , the choice of the top size interval in this case has to be little arbitrary . This , however , does not affect the accuracy of the computations , since the size distribution over the entire size range is uniquely described by Eq. (2.59) , and is independent of the choice of the top size interval .

The computed size distribution curves at the start of the experiment and at t = 9 minutes are shown in Figure 6.1 . The variation of the inner breakage parameter with grinding time for four size intervals i = 1, 3, 6 and 10 has been shown in Figure 6.2 . It can be seen that , at least , in case of coarse size intervals , the variation is quite significant . Moreover , the value of $b_{i,i}$ for coarse size intervals is markedly different than that for the same in the Bass model .

6.3 The SSF Model

In this model we formulate a separable selekage function (SSF) as follows

$$L(v,x) = L_1(v)\, L_2(x) \qquad (6.26)$$

where $L_1(v)$ is some function of particle size v,

It can be shown that in case of the separable

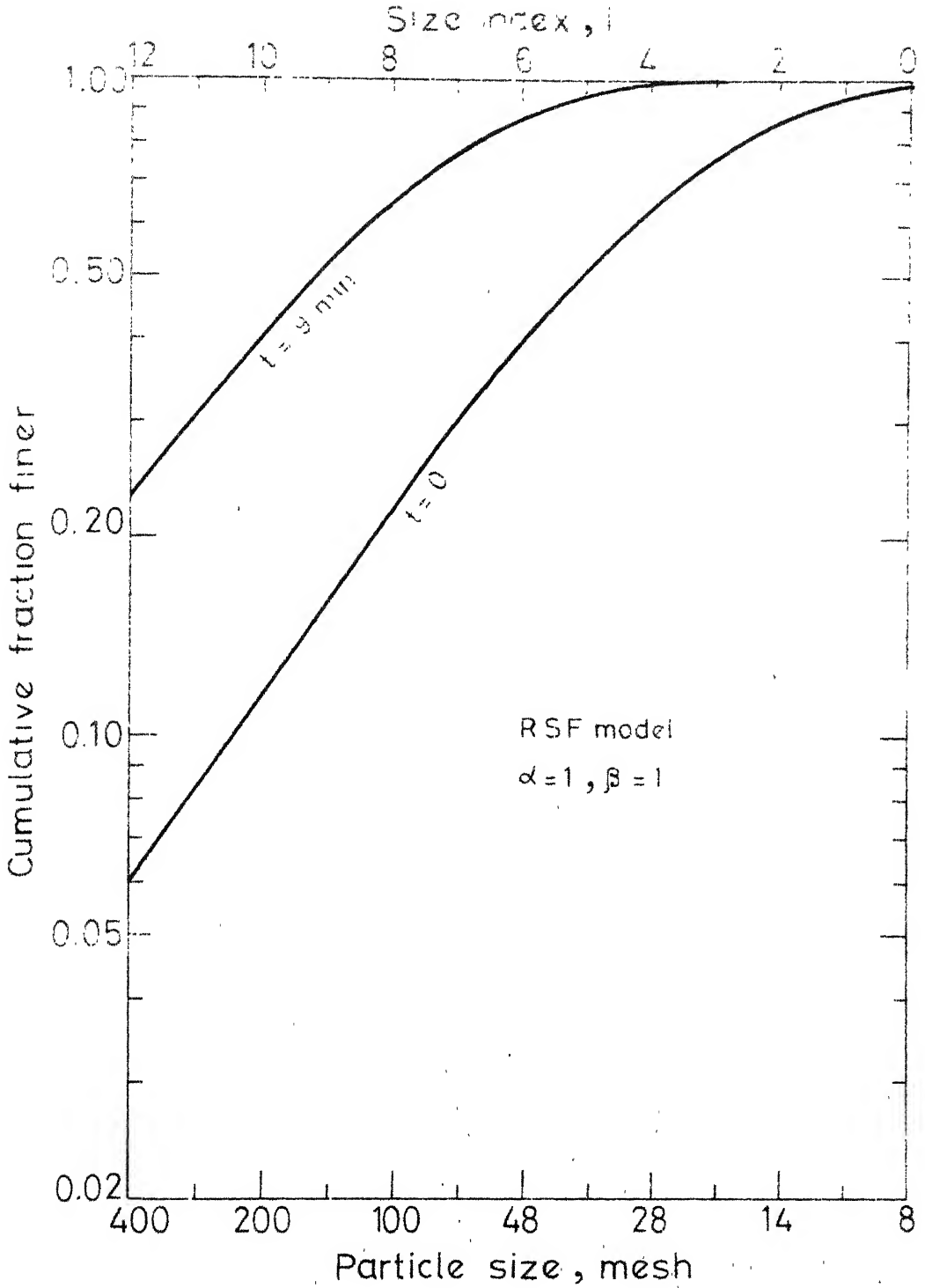

Fig. 6.1 Initial and final particle size distributions in simulation experiment for the RSF model, $\alpha = 1, \beta = 1$

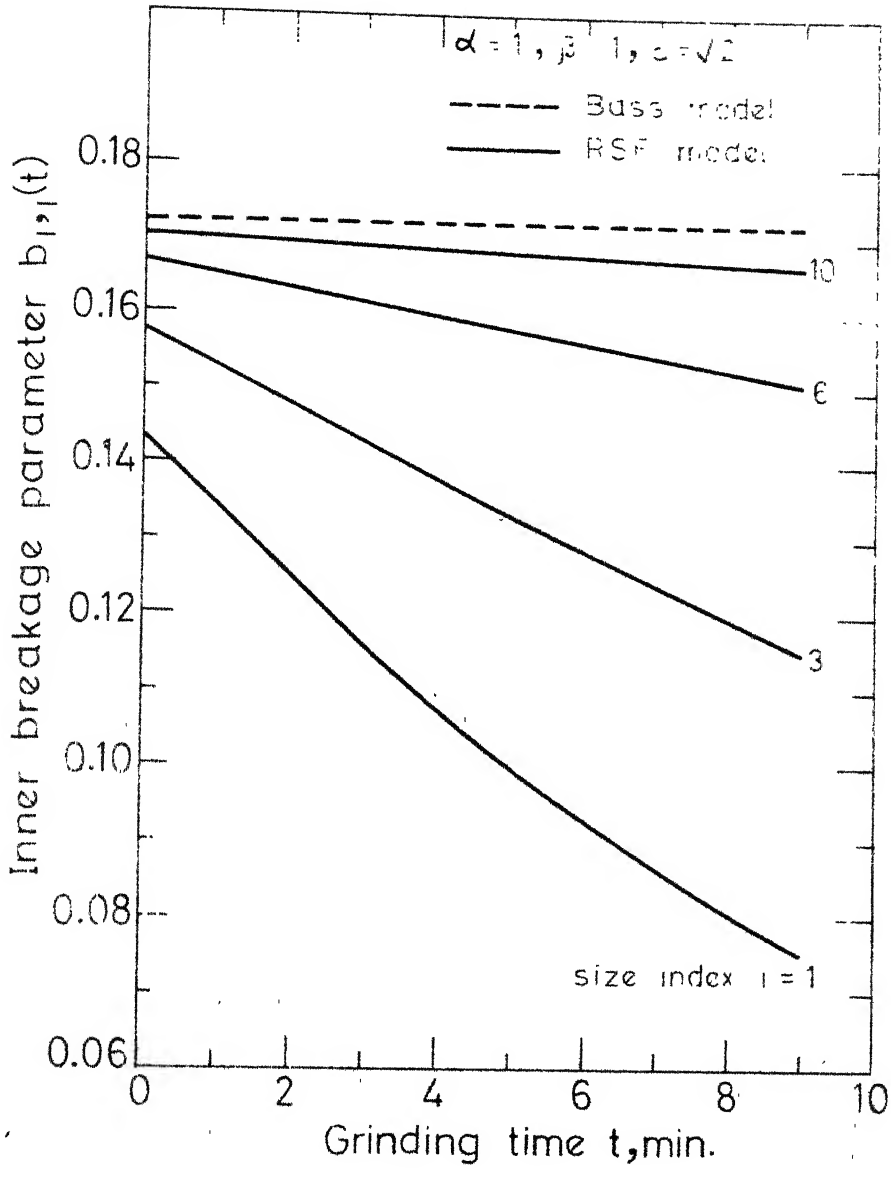

Fig. 6.2 Time variation of inner breakage parameter in the RSF model for $\alpha = 1$, $\beta = 1$, $\delta = \sqrt{2}$

selekage function the form is necessarily

$$L(v,x) = h(v) \frac{L_2(x)}{L_2(v)} \tag{6.27}$$

where $h(x)$ is any positive definite function over the range $0 < x < x_o$. It follows that

$$L_1(v) = \frac{h(v)}{L_2(v)} \tag{6.28}$$

and from Eqs. (5.21) , (5.22) and (6.27)

$$S(v) = h(v)$$

and

$$\bar{B}(v,x) = \frac{L_2(x)}{L_2(v)} \tag{6.29}$$

It should be noted that in contrast to the RSF model , in this case $S(0)$ need not necessarily be equal to zero and $S(x)$ need not necessarily be a monotonically non-decreasing function .

The discrete size grinding parameters in Eqs. (5.27) and (5.28) become

$$S_i(t) = \frac{L_2(x_i) \int_{x_i}^{x_{i-1}} L_1(v)\, M(v,t)\, dv}{\int_{x_i}^{x_{i-1}} M(v,t)\, dv} \tag{6.30}$$

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$$L(v,x) = h(v) \frac{L_2(x)}{L_2(v)} \tag{6.27}$$

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The discrete size grinding parameters in Eqs. (5.27) and (5.28) become

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Or, compactly

$$S_i(t) = L_2(x_i)\, P(x_i, t) \tag{6.31}$$

where

$$P(x_i t) = \frac{\int_{x_i}^{x_{i-1}} L_1(v)\, M(v,t)\, dv}{\int_{x_i}^{x_{i-1}} M(v,t)\, dv} \tag{6.32}$$

Rate parameter is therefore , a function of time . And

$$B_{i,j} = \frac{L_2(x_{i-1}) - L_2(x_i)}{L_2(x_j)} \tag{6.33}$$

which is independent of time and similar to $B_{i,j}$ in the RSF model in Eq. (6.22) . Substitutions in the master equation Eq. (5.10) generate the size discretized SSF model for grinding kinetics

$$\frac{dM_i(t)}{dt} = -L_2(x_i)\, P(x_i,t)\, M_i(t) + [L_2(x_{i-1}) - L_2(x_i)] \sum_{j=1}^{i-1} P(x_j ;t)\, M_j(t) \tag{6.34}$$

Incidentally, this is the first time this model has been clearly delineated . The inner breakage parameter for the SSF model is obtained from Eq. (5.30)

$$b_{i,i}(t) = 1 - \frac{L_2(x_i) \int_{x_i}^{x_{i-1}} L_1(v)\, M(v,t)\, dv}{\int_{x_i}^{x_{i-1}} S(v)\, M(v,t)\, dv} \qquad (6.35)$$

The selekage function in Eq. (6.13), which is the product of the selection and breakage functions given in Eqs. (2.56) and (2.57), respectively, constitutes a valid example of separable selekage function . In order to compute the time variation of the rate parameter in the SSF model arising from the discretization step and also the time variation of the inner breakage parameter, we substitute Eq. (2.59) into Eqs. (6.30) and (6.35) , and obtain the following explicit expressions for $S_i(t)$ and $b_{i,i}(t)$

$$S_i(t) = \frac{A\, x_i^{\beta} \int_{x_i}^{x_{i-1}} v^{\alpha-1} \exp\left[-\frac{1}{\alpha h}\left(\frac{v}{\mu_1(t)}\right)^{\alpha}\right] dv}{\int_{x_i}^{x_{i-1}} v^{\beta-1} \exp\left[-\frac{1}{\alpha h}\left(\frac{v}{\mu_1(t)}\right)^{\alpha}\right] dv} \qquad (6.36)$$

and

$$b_{i,i}(t) = 1 - \frac{x_i^{\beta} \int_{x_i}^{x_{i-1}} v^{\alpha-1} \exp\left[-\frac{1}{\alpha h}\left(\frac{v}{\mu_1(t)}\right)^{\alpha}\right] dv}{\int_{x_i}^{x_{i-1}} v^{\alpha+\beta-1} \exp\left[-\frac{1}{\alpha h}\left(\frac{v}{\mu_1(t)}\right)^{\alpha}\right] dv} \qquad (6.37)$$

For simulation, the values of x_0 , A , and $\mu_1(0)$ were kept the same as in the previous section . Two combinations of α and β were chosen - (1) $\alpha = 0.5$ and $\beta = 2$ and (ii) $\alpha = 1.5$ and $\beta = 0.5$. In the first simulation experiment, computations were carried out for $\delta = \sqrt{2}$ as well as $\delta = \sqrt[4]{2}$. The particle size distribution curves at the start of the experiment i.e., at t = 0 and at the end i.e., at t = 9 minutes, are shown in Figure 6.3 . The results of the variation in $S_i(t)$ as a function of grinding time are shown in Figures 6.4 and 6.5 and those of the time variation in $b_{i,i}(t)$ are shown in Figures 6.6 and 6.7 . The rate parameter values have been normalized to $S_i(t)/S_i(0)$. It can be seen from these figures that the change is maximum for size index i=1 and decreases progressively with increasing size index . This trend is quite expected because the particle size distribution in the coarse size range changes with time much faster than the size distribution in the fine size range . Another significant observation is that the time variation of discrete parameters is comparatively low when narrower sieve

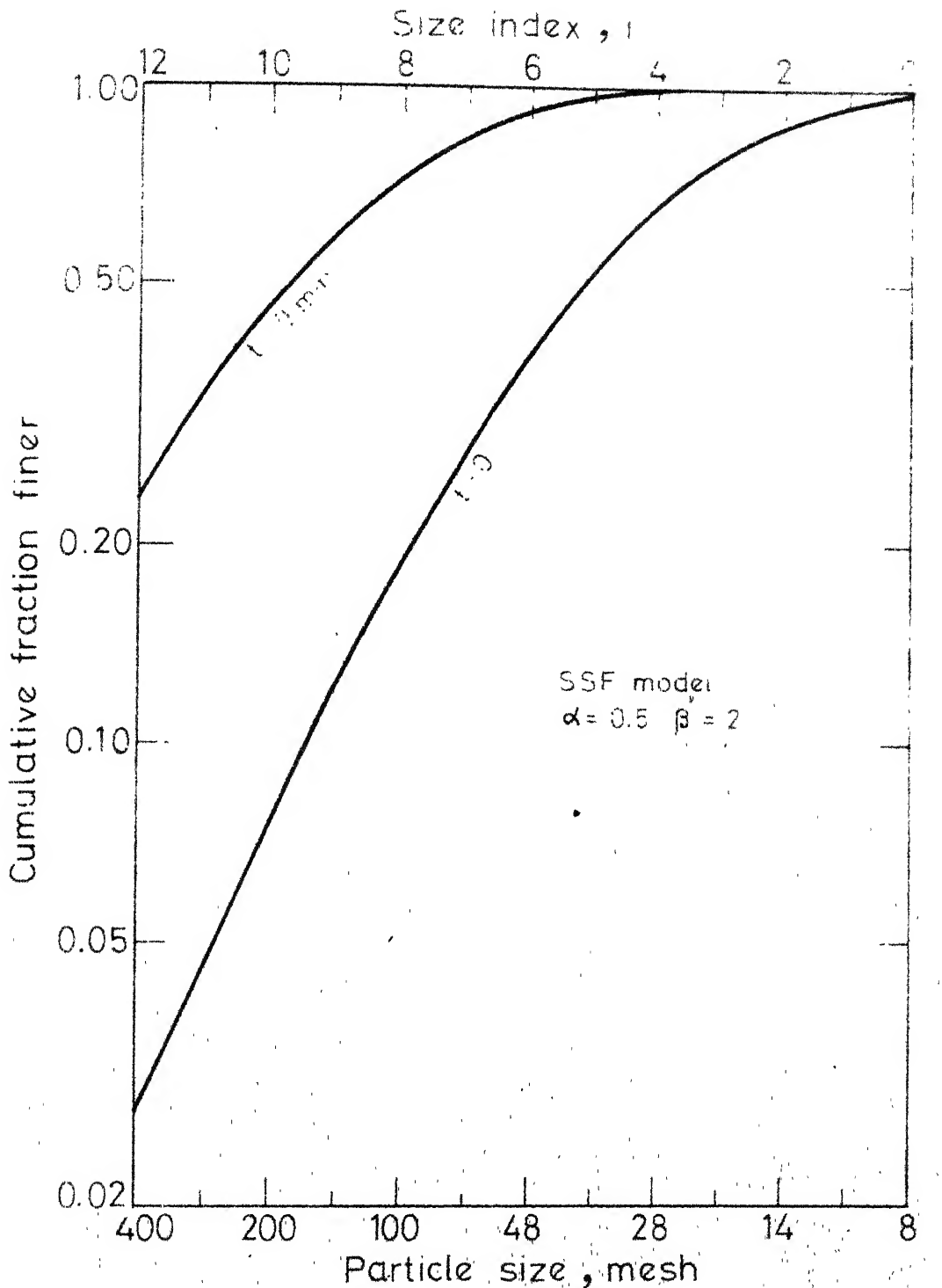


Fig. 6.3 Initial and final particle size distributions in simulation experiment for the SSF model, $\alpha = 0.5$ $\beta = 2$

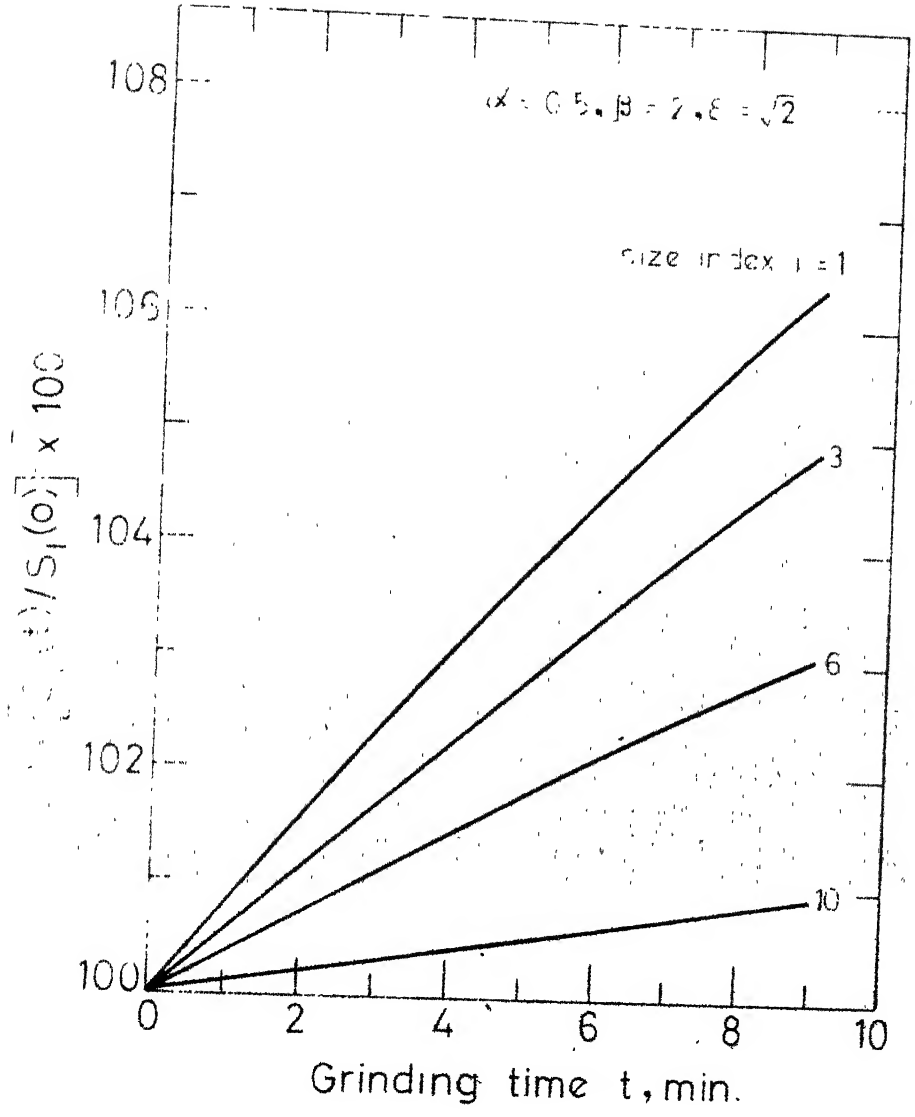


Fig. 6.4 Time variation of rate parameter in the SSF model for $\alpha = 0.5$, $\beta = 2$ and $\delta = \sqrt{2}$

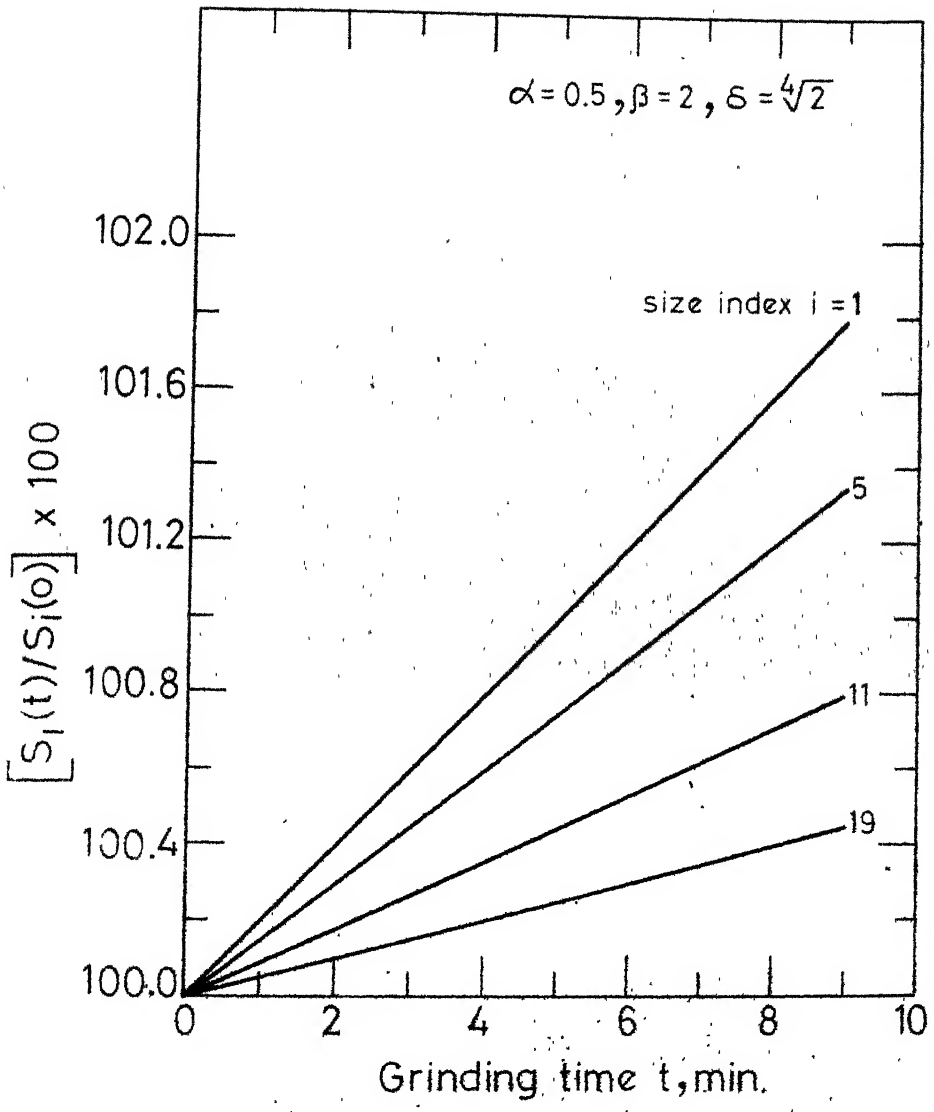


Fig. 6.5 Time variation of rate parameter in the SSF model for $\alpha = 0.5, \beta = 2$ and $\delta = \sqrt[4]{2}$.

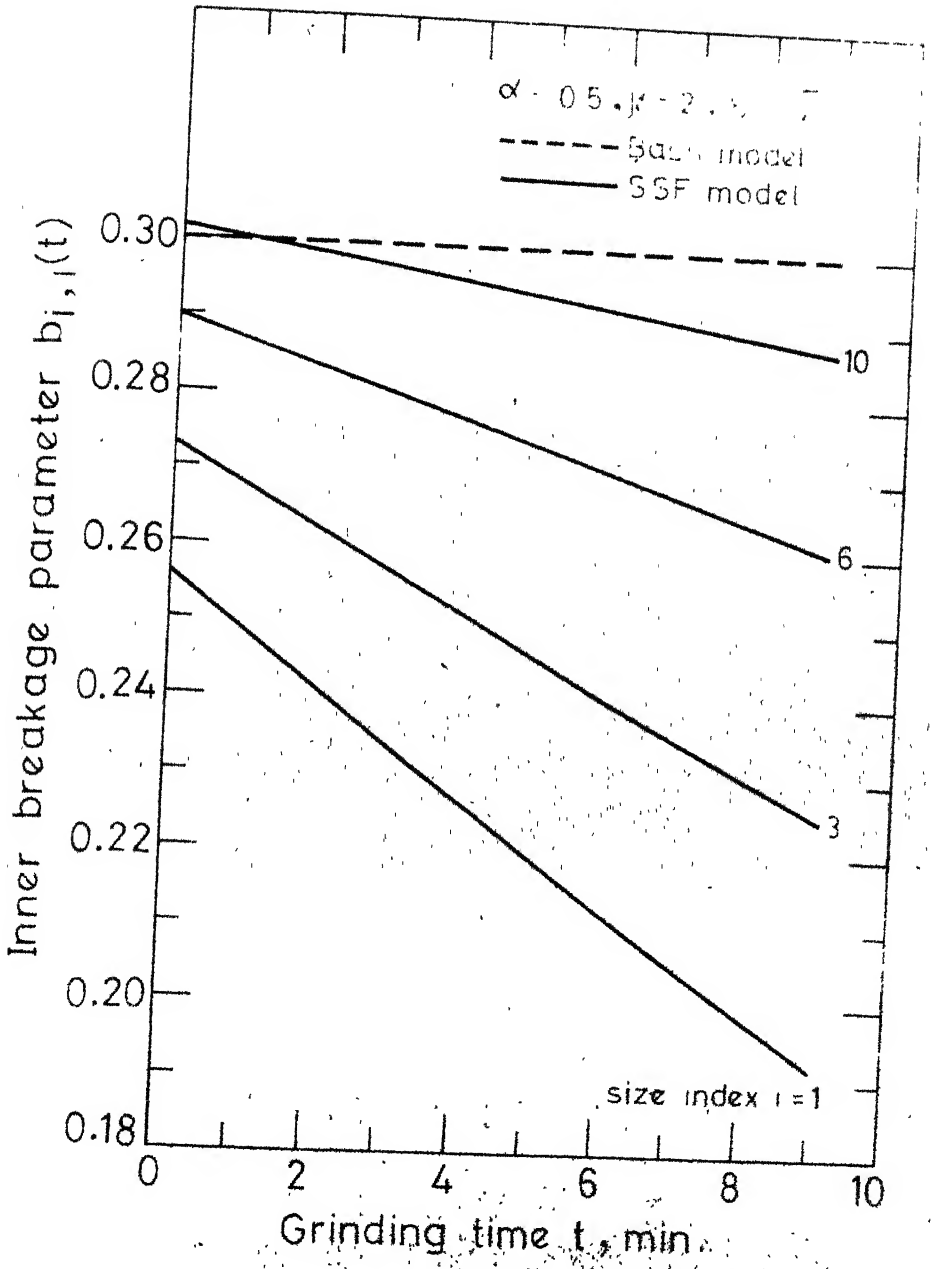


Fig.6.6 Time variation of inner breakage parameter in the SSF model for $\alpha = 0.5, \beta = 2$ and $\delta = \sqrt{2}$

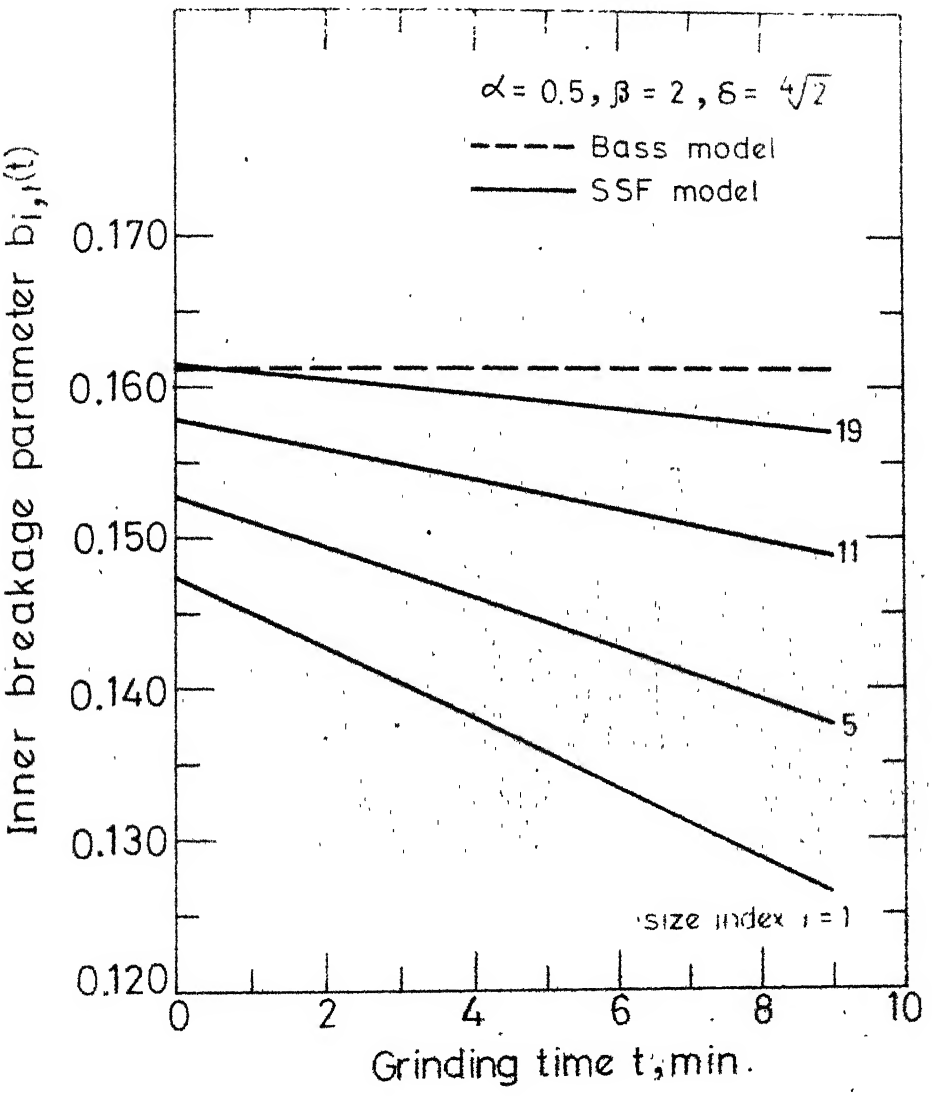


Fig.6.7 Time variation of inner breakage parameter in the SSF model for $\alpha = 0.5, \beta = 2$ and $\delta = 4\sqrt{2}$

size intervals are used for describing the particle size distribution.

It should be pointed out that , in practice, more pronounced variations may be expected to occur because, unlike in this example, the feed size distribution curve in the coarse size range does not normally exhibit a pronounced curvature . For illustration of this point, the values of the time invariant rate parameters (uniformly distributed particle size distribution - Bass model) and those in the above simulation experiment at t = 9 minutes have been compared in Table 6.2 . The percent change in the values of rate parameters is now almost two times of that shown in Figures 6.4 and 6.5 . Many other simulation experiments with different combinations of α and β showed that if the difference in the values of α and β is increased, time variation of discrete size parameters becomes more significant . In one of such experiments with $\alpha = 0.5$ and $\beta = 4$, the per-cent increase in the values of the rate parameters was found to be approximately twice of that reported for $\alpha = 0.5$ and $\beta = 2$.

The initial and final particle size distribution curves for $\alpha = 1.5$ and $\beta = 0.5$ are shown in Figure 6.8 . The time variation of the rate and inner breakage parameters

TABLE - 6.2

Comparison of the values of time invariant rate parameters and the rate parameters at t = 9 min. in the simulation experiment (Fig. 6.3, $\alpha = 0.5$, $\beta = 2$).

$\delta = \sqrt{2}$				$\delta = \sqrt[4]{2}$			
i	S_i (time inv.)	S_i (t=9 min.)	Percent difference in S_i	i	S_i (time inv.)	S_i (t=9 min.)	Percent difference in S_i
1	0.7682	0.8547	11.2	1	0.9531	0.9863	3.5
3	0.5432	0.5861	7.9	5	0.6739	0.6908	2.5
6	0.3230	0.3365	4.2	11	0.4001	0.4067	1.7
10	0.1615	0.1637	1.4	19	0.2000	0.2019	0.95

is shown in Figures 6.9 and 6.10 , respectively . In contrast to the Figures 6.4 and 6.6 where rate parameters increase with grinding time, Figure 6.9 shows that the rate parameters constantly decrease with increasing grinding time . In fact, simulation experiments showed that this is generally true for all the cases where α_i is greater than β . Similarly , for $\alpha < \beta$, the rate parameters were always found to increase with increasing grinding time . This observation can be explained in the following manner . For the case $\alpha > \beta$, $L_1(v)$ is a monotonically increasing function in the particle size v . As grinding proceeds, the relative mass density of particles $[\, M(v,t)/M_i(t) \,]$ decreases for the coarser particles and it increases for the finer ones . It, therefore, follows from Eq. (6.30) that due to the increasing proportion of finer particles in the discrete size interval, whose contribution to the rate parameter (i.e., $L_1(v)$) is relatively small, the rate parameter decreases with increasing grinding time . Similarly, it can be argued for the case $\alpha < \beta$ that the rate parameter always increases with increasing grinding time . It will be further seen, that the logarithmic - linear plots of $M_i(t)$ vs. t for a single size material always tend to concave downwards for $\alpha < \beta$ and they concave upwards for $\alpha > \beta$. Examples of both the cases can be found in the

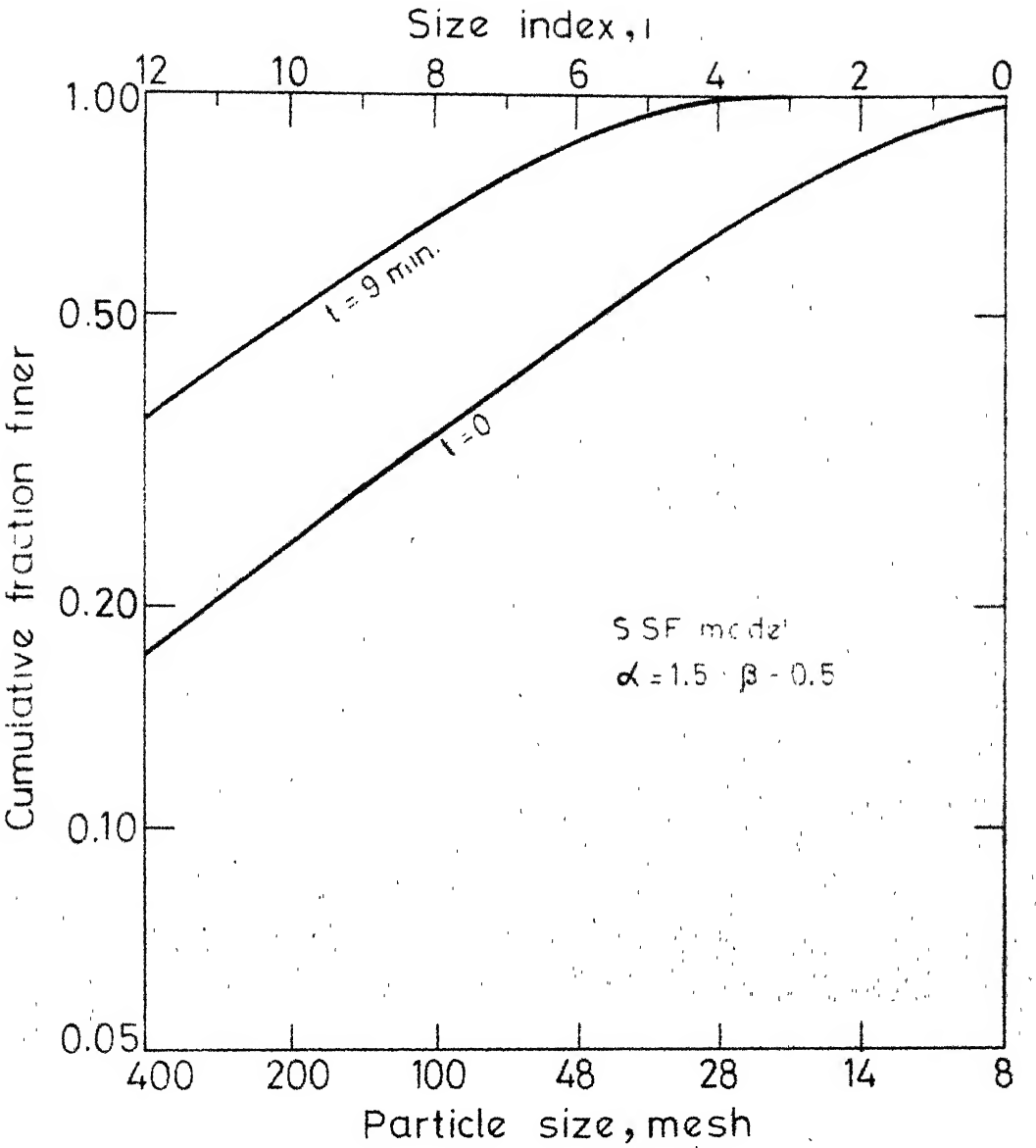


Fig. 6.8 Initial and final particle size distributions in simulation experiment for the SSF model, $\alpha = 1.5$, $\beta = 0.5$

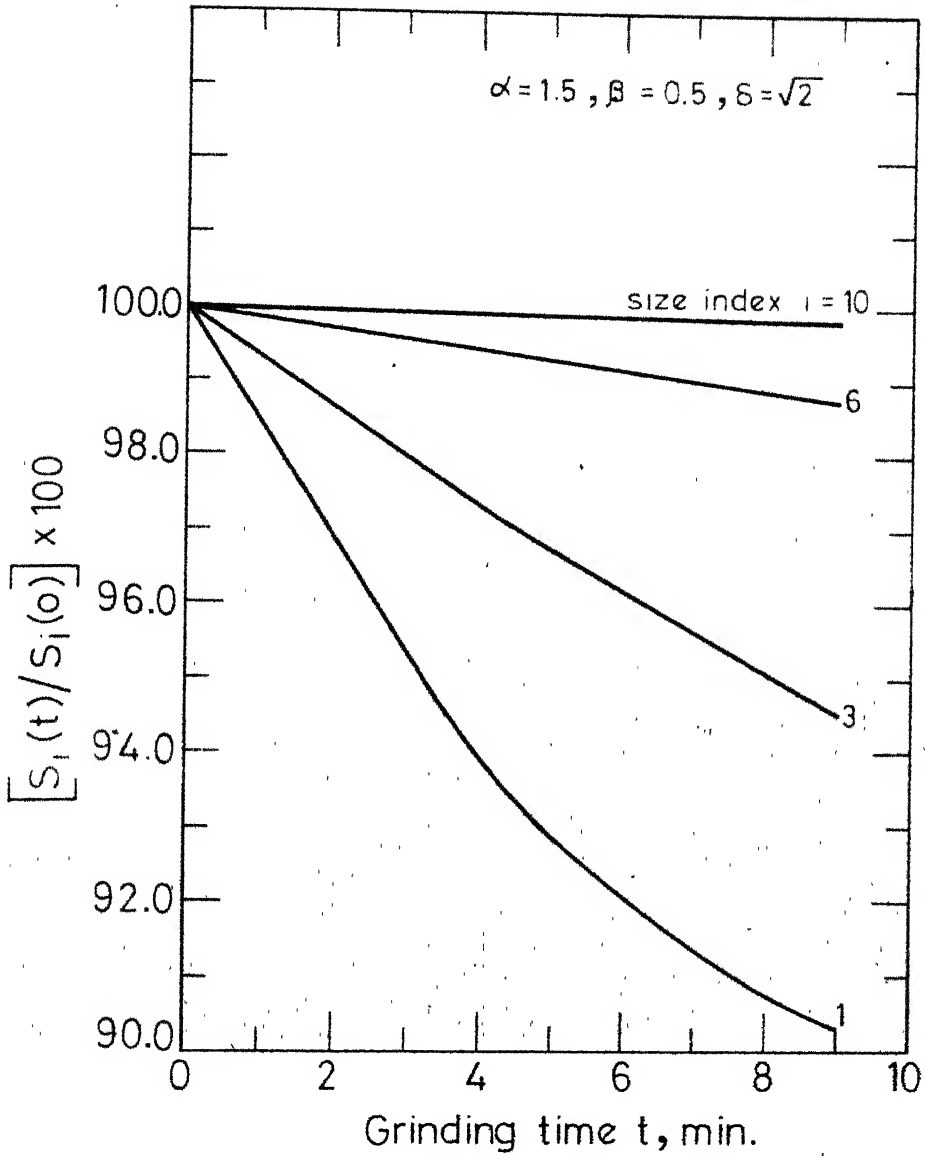


Fig. 6 9 Time variation of rate parameter in the SSF model for $\alpha = 1.5$, $\beta = 0.5$ and $\delta = \sqrt{2}$

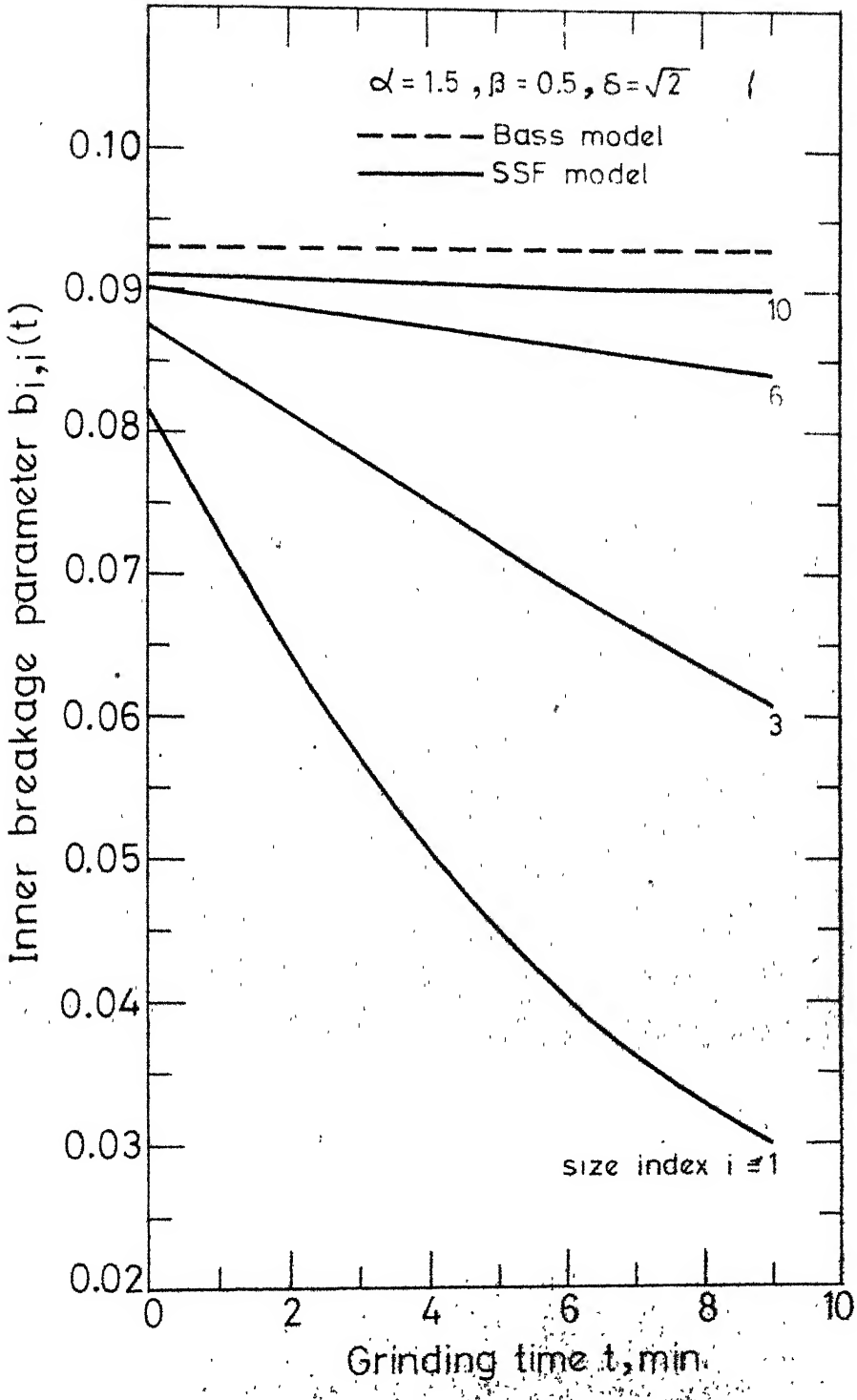


Fig. 6.10 Time variation of inner breakage parameter in the SSF model for $\alpha = 1.5$, $\beta = 0.5$, and $\delta = \sqrt{2}$

grinding literature [41,82,84] .

Finally, in view of the fact that the time variation is more pronounced in case of the inner breakage parameters than that observed for the rate parameters, it may be suggested that of the two discrete size grinding models, the model 'A' is a better choice, if the time invariant parameter models are to be used .

CHAPTER 7

ANALYSIS OF SOME SPECIAL GRINDING PHENOMENA

In light of the three specialized size discretized models available to us, in the present chapter an attempt is made to seek unambiguous answers to the problems concerning the occurence of various special grinding phenomena that have been observed empirically. In particular , we shall examine- (i) first order disappearance kinetics for a discrete size interval, (ii) zero order production of fines, (iii) Rosin-Rammler kinetics, (iv) Alyavdin kinetics, (v) difference-similar breakage parameters and (vi) size variation of rate parameters according to a power law.

7.1 First Order Disappearance Kinetics for a Discrete Size Interval

In the literature, there has been an implicit or explicit assumption in treatment of discrete size grinding models to the effect that if the material breaks in the manner of a first order kinetic law, then the rate of disappearance of solids from a discrete size interval will also conform to the same law | 72 | . Sometimes, it is stated that this is a approximate hypothesis but, as Harris | 45 | has pointed out, the approximations which lead to these

kinetics for discrete size intervals have never been clearly defined. Examination of Eq. (5.26) shows that the expression for the pure disappearance kinetics is

$$\frac{dM_i(t)}{dt} = - \int_{x_i}^{x_{i-1}} L(v,x_i)\, M(v,t)\, dv \qquad (7.1)$$

It can be seen from Eq. (7.1) that the rate of change of $M_i(t)$ cannot be explicitly expressed as a function of $M_i(t)$ only. Thus, in general, it is not possible to realize any order kinetics, in terms of $M_i(t)$, leave aside the first order law . However it will be seen that the restrictions on the particle mass density function $M(x,t)$ in Eqs. (6.1) and (6.2) , resulting in the Bass model, and/or the restriction on the selekage function in Eq. (6.15), resulting in the RSF model, are sufficient conditions for first order hypothesis to prevail. But it should be noted that the rate parameters in these two models, given in Eqs. (6.3) and (6.21), have completely different forms and structures.

It should be pointed out that the first condition of uniform size distribution within individual size intervals can be met by only extremely narrow size intervals, which cannot be satisfied in practice . The second condition which though represents a highly unlikely situation, provides important clues to this problem. It will be seen from

Eq. (5.27) that lesser is the dependence of function L on the parent particle size v, greater is the size interval width that can be afforded in practice to obtain a reasonable first-order- kinetics in the conventional sense. Of course, when the selekage function is completely independent of the size of the parent particle, the width of the discrete size interval is immaterial.

Austin | 28 | has pointed out that when a discrete size interval is too wide, the conventional first-order disappearance kinetics plots are either concave upwards or concave downwards. He has ascribed this phenomenon to a number of factors, which have been mentioned earlier in Chapter 2. A close inspection of Eq. (5.27) for the rate parameter S_i , however, reveals that the answer to this problem lies, in fact, with the Selekage function L(v,x). For a discrete size fraction i, $L(v,x_i)$, which represents the fraction of material of size v which falls out of size interval i during the breakage, is normally expected to be either a monotonically increasing function in v or a monotonically decreasing function in v , over the size range $x_i < v < x_{i-1}$. Moreover, as the grinding proceeds, the proportion of coarser particles in the discrete size interval decreases and that of the finer ones increases. Hence, it is clear from Eq. (5.27) that when $L(v,x_i)$ is a monotonically

increasing function in v, lesser proportion of material of size interval i will break out of it at higher grinding times. Hence, S_i will exhibit a decreasing trend with increasing grinding time, and the disappearance kinetics plot will therefore, concave upwards. The reverse holds when $L(v,x_i)$ is a monotonically decrrasing function in particle size v. Obviously, depending on the initial size distribution of the particles, and the nature of the function $L(v,x_i)$ over the size interval $x_i < v < x_{i-1}$, even more complex kinetics may be observed.

Finally, the other factors (Chapter 2.5) mentioned by Austin | 28 | , may augment or nullify the effects of initial particle size distribution and the selekage function on the disappearance kinetics . However, in view of the results obtained on time variation of the rate parameters in Chapter 6 and those reported later in Chapter 9, it would seem that these factors are of secondary importance and the main determining factor is the selekage function.

7.2 Zero Order Production of Fines

We let $x = x_i$, and rewrite Eq. (2.34) as follows

$$\frac{dF_i(t)}{dt} = \int_{x_i}^{x_0} L(v,x_i) \frac{\partial F(v,t)}{\partial v} \, dv \qquad (7.2)$$

The phenomenon of the zero order production of fines requires that right side of the equality in Eq. (7.2) be a constant, at least for some initial period of time. We shall now scrutinise the discrete size models available to us for possible justification for this phenomenon.

In case of the Bass model we first write Eq. (7.2) as

$$\frac{dF_i(t)}{dt} = \sum_{j=1}^{i} \int_{x_j}^{x_{j-1}} \left[\int_{0}^{x_i} S(v)\, B(v,x)\, dx \right] M(v,t)\, dv \qquad (7.3)$$

and then invoke the stipulation of uniform distribution within the individual size intervals to obtain

$$\frac{dF_i(t)}{dt} = \sum_{j=1}^{i} \frac{M_j(t)}{x_{j-1} - x_j} \int_{0}^{x_i} \int_{x_j}^{x_{j-1}} S(v)\, B(v,x)\, dv\, dx \qquad (7.4)$$

or

$$\frac{dF_i(t)}{dt} = \sum_{j=1}^{i} \frac{M_j(t)}{x_{j-1} - x_j} \sum_{k \geq i+1} \int_{x_k}^{x_{k-1}} \int_{x_j}^{x_{j-1}} S(v) B(v,x)\, dv\, dx \qquad (7.5)$$

Substitution of Eq. (2.40) gives

$$\frac{dF_i(t)}{dt} = \sum_{j=1}^{i} \left[M_j(t) \sum_{k \geq i+1} v_{k,j} \right] \qquad (7.6)$$

For the RSF model , we substitute Eq. (6.16) into Eq. (7.1) and recall that $F(x_0, t) = 1$ for $t \geq 0$, hence

we have

$$\frac{dF_i(t)}{dt} = L_2(x_i) \left[1 - F_i(t) \right] \tag{7.7}$$

For the SSF model, we substitute Eq. (6.26) into Eq. (7.1) and obtain the following expression

$$\frac{dF_i(t)}{dt} = L_2(x_i) \int_{x_i}^{x_0} L_1(v)\, M(v,t)\, dv \tag{7.8}$$

which we write as

$$\frac{dF_i(t)}{dt} = L_2(x_i) \sum_{j=1}^{i} \int_{x_j}^{x_{j-1}} L_1(v)\, M(v,t)\, dv \tag{7.9}$$

Substitution of Eq. (6.30) followed by substitution of Eq. (6.33) gives

$$\frac{dF_i(t)}{dt} = \sum_{j=1}^{i} S_j(t)\, \bar{B}_{i,j}\, M_j(t) \tag{7.10}$$

Equation (7.10) is apparently same as the one presented by Herbst and Fuerstenau | 58 | for the rate of production of fines. However, it should be noted that in contrast to their assumptions of negligible inner breakage parameter and environment dependent rate parameter, we have derived Eq.(7.10) by discretization of an inherently linear grinding model; the time dependent rate and time-invariant breakage parameters are the characteristics of the SSF model.

Inspection of the three expressions in Eqs. (7.6), (7.7 and (7.10) for the rate of production of fines shows that the most natural candidate for the realization of the zero order production phenomenon is the RSF model. We argue that if x_i is much smaller than the smallest particles in the feed then, at least for initial period, $F_i(t) << 1$, and

$$\frac{dF_i(t)}{dt} \simeq L_2(x_i) \tag{7.11}$$

Comparing Eq. (7.11) with Eq. (2.116) and (2.124), it can be seen that the function φ is nothing but the function L_2. Substitution of Eq. (6.21) into Eq. (7.11) gives

$$\frac{dF_i(t)}{dt} \simeq S_i \tag{7.12}$$

Substitution of Eq. (6.16) in Eq. (5.29) gives for the RSF model

$$\bar{B}_{i,j} = \frac{S_i}{S_j} \tag{7.13}$$

From which it follows that

$$S_i = S_j \bar{B}_{i,j} \tag{7.14}$$

Combining Eq. (7.14) with Eq. (7.12) gives the restrictive interrelationship between the rate and breakage parameters proposed by Mika et al. | 42 |, and Herbst and Fuerstenau |58| as a sufficient condition for zero order production of fines

$$\frac{dF_i(t)}{dt} = S_j \bar{B}_{i,j} \tag{7.15}$$

This condition is then a consequence of the RSF model.

It should be pointed out that only when we specialize the function $L_2(x) = A\, x^{\alpha}$, the Eq. (7.12) becomes

$$\frac{dF_i(t)}{dt} \simeq A\, x_i^{\alpha} \tag{7.16}$$

which is the form observed by Arbiter and Bhrany | 83 | . Since there is no reason why $L_2(x)$ should always be of the form $A\, x^{\alpha}$, for this reason, it is possible that in certain grinding systems the zero order production rate constants ϱ_i may not conform to the form x_i^{α}.

7.3 The Rosin-Rammler Grinding Kinetics

Starting from Eq. (2.106) we shall derive an exact solution for the RSF model. Substitutions of Eqs.(6.21) and (7.14) causes the summation term to vanish, and a straight forward integration of the remaining equation gives

$$R_i(t) = R_i(0) \exp \left[- L_2(x_i)\, t \right] . \tag{7.17}$$

This is the discrete size analog of the continuous size Rosin-Rammler type distribution in Eq. (2.54). When the reduced selekage function is equal to $A\, x^{\alpha}$, we have the Rosin Rammler distribution

$$R_i(t) = R_i(0) \exp \left[- A\, x_i^{\alpha}\, t \right] \tag{7.18}$$

We like to stress that the RSF model holds when a plot of ln $R_i(t)$ against time t is a straight line. On the other hand, a plot of ln ln $[R_i(0) / R_i(t)]$ against ln x_i will be a straight line (with slope equal to α) only if $L_2(x) = A x^{\alpha}$.

Over the years, the comminutionists have developed an uncertain attitude towards the Rosin - Rammler distribution. Many workers feel that by virtue of its relatively simple form, the Rosin-Rammler representation of the grinding kinetics cannot conceivably do justice to this complex system. Nonetheless, the Rosin-Rammler kinetics continues to attract a steady patronage, either overtly, as for example, by Austin and Luckie | 105 |, Furuya et al. | 106 |, and Gupta and Kapur | 19 | , or covertly, as for example, by Schonert | 36 | , and Herbst and Fuerstenau | 58 | .

7.4 The Alyavdin Kinetics

Harris and Chakravarti | 46 | have shown that the Alyavdin size distribution function fits a variety of grinding data quite well, specially where the Rosin-Rammler distribution turns out to be inadequate . The Alyavdin distribution is

$$R(x,t) = R(x,0) \exp (\psi(x) t^{1+a}), \quad a \neq 0 \qquad (7.19)$$

where ψ is a function of particle size, and the exponent a is nominally a constant, although it may exhibit a weak dependence on x and/or t . Harris | 38 | found it convenient to introduce the exponent a where departure from the Rosin-Rammler type kinetics occurs . His derivation essentially requires that the selekage function be written as

$$L_2(v,x,t) = t^a L_2(x) \tag{7.20}$$

We shall present simple arguments for finding an approximate solution to the SSF model. From Eqs. (5.29), (6.26) and (6.31)

$$S_j(t)\ \bar{B}_{i,j} = S_i(t)\ \frac{P(x_j,\ t)}{P(x_i,\ t)} \tag{7.21}$$

Substitution in Eq. (2.106) gives

$$\frac{dR_i(t)}{S_i(t)\ dt} = -\ R_i(t) + \sum_{j=1}^{i-1} \left[\ \frac{P(x_{j+1},t) - P(x_j,t)}{P(x_i,\ t)}\ \right] R_j(t) \tag{7.22}$$

If the function P is only weakly dependent on x such that the following inequality holds

$$\sum_{j=1}^{i-1} \left[\left(\frac{P(x_{j+1}\ ,\ t) - P(x_j\ ,\ t)}{P(x_i\ ,\ t)} \right) \frac{R_j(t)}{R_i(t)}\ \right] \ll 1 \tag{7.23}$$

then to a first approximation we may neglect the summation term, and solution to Eq. (7.22) is given by

$$R_i(t) = R_i(0)\ \exp\left[-\ L_2(x_i) \int_0^t P(x_i,t')\ dt' \right] \tag{7.24}$$

where the integral term is a weak function of x . Equation (7.24) is the general form of the Alyavdin distribution in Eq. (7.19) . Simulation studies have shown that the R(x,t) values computed from the similarity solutioh in Eq.(2.59) for separable selekage function in Eq. (6.13) (with α close to β) , can be plotted as per the Alyavdin equation and, in most instances, the deviations are well with in the experimental errors in the data (Figure 7.1). The point to be stressed is that the Alyavdin distribution need not arise from any inherent time - dependence of the selection and breakage functions, as in Eq. (2.63) or Eq. (7.20), but it could well be simply an approximate solution to the integro differential equation of grinding in cumulative fraction retained mode for a class of time-invariant selekage functions, as the one given in Eq. (6.13) .

7.5 Difference - Similar Breakage Parameters

For cumulative breakage parameter $\bar{B}_{i,j}$, the difference similar characteristics imply that

$$\bar{B}_{i,j} = \bar{B}_{i-j} \qquad (7.25)$$

Inspection of Eqs. (5.28) and (5.29) indicates, however, that, in general, the breakage parameters vary with time . Furthermore, even if the cumulative breakage function is

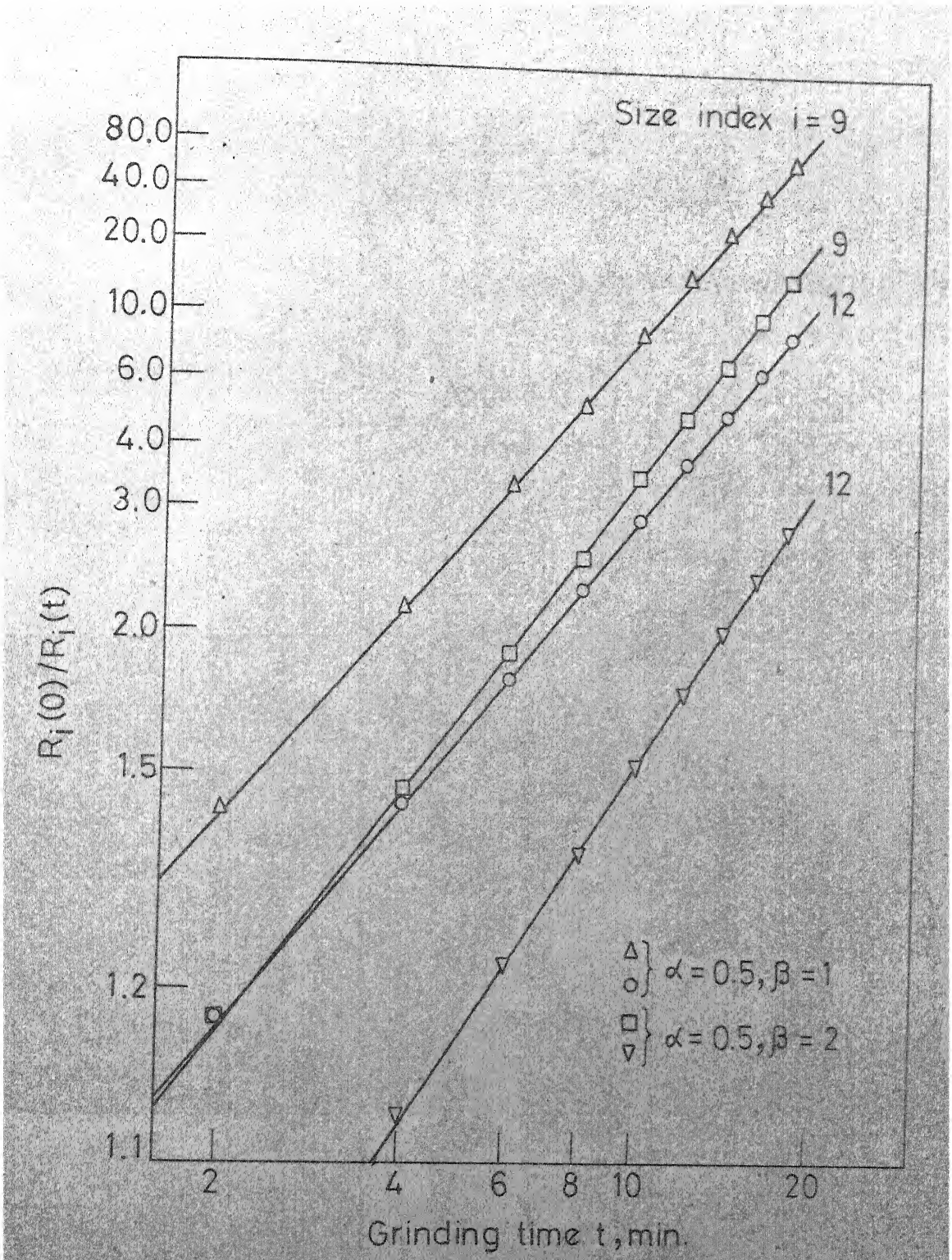


Fig.7.1 Alyavdin-Weibull plots of synthetic particle size distribution data generated from Eq.(2.59) for SSF model, $L(v,x) = Av^{\alpha-\beta} x^{\beta}$.

normalizable in following manner

$$\bar{B}(v,x) = \bar{B}(x/v) \tag{7.26}$$

in the general case we are unable to attribute normalizability i.e. the difference similar form to the cumulative breakage parameter.

From Eq. (5.29) the expressions for the cumulative breakage parameter in three discrete size models delineated previously are as follows. For the Bass model

$$\bar{B}_{i,j} = \frac{\int_{x_j}^{x_{j-1}} L(v,x_i)\, dv}{\int_{x_j}^{x_{j-1}} L(v,x_j)\, dv} \tag{7.27}$$

and for the RSF and SSF models

$$\bar{B}_{i,j} = \frac{L_2(x_i)}{L_2(x_j)} \tag{7.28}$$

In case of the Bass model, if and only if, the selection and breakage functions are given by

$$S(v) = A\, v^{\alpha} \tag{2.56}$$

$$\bar{B}(v,x) = \bar{B}(x/v) \tag{7.26}$$

the breakage parameter in Eq. (7.27) is difference-similar (normalizable) as follows

if and only if, the selection and breakage functions are given by Eqs. (2.56) and (7.26), the rate parameter is given by following power law

$$S_i = \frac{A\, x_i^{\alpha}}{\delta - 1} \quad \phi\ (1\ ,\ \delta^{-1}) \tag{7.33}$$

Similarly, in case of RSF model, if and only if, the selekage function is given by $L(v,x) = A\, x^{\alpha}$, the rate parameter obeys the power law and the expression from Eq. (6.21) is

$$S_i = A\, x_i^{\alpha} \tag{7.34}$$

It should be pointed out that except for SSF model, the phenomenon of difference - similar breakage parameter is always coupled with the phenomenon of a power law for size variation of the rate parameter.

CHAPTER 8

IDENTIFICATION OF THE GRINDING SYSTEM IN TERMS OF A MILL FUNCTION

As pointed out earlier in Chapter 2 , there does not exist at present a practical and reliable method for estimation of the two basic functions $S(x)$ and $B(v,x)$, which uniquely characterize the mill-material grinding system . Although , a number of approximate methods have been developed for the estimation of the size discretized parameters , a unique identification of the grinding system is still not possible because of the dependence of these parameters on the particle size spectra . It may then be concluded that at present no convenient scheme is available for a true identification of the grinding system . In this chapter , a novel approach is presented for identification in terms of a mill function , which is based on a continuous size and discrete time representation of the grinding system . In the next chapter this approach has been further extended for estimation of the size discretized rate and breakage parameters in the model 'A' in Eq. (2.79) .

8.1 Formulation of the Grinding Model

It has been recognised that there exist three mechanisms for size reduction of particles , namely , impact fracture , chipping and abrasion | 107 | . There are unfortunately conceptual difficulties in arriving at unambigious definitions of these mechanisms . We prefer to define impact fracture as catastrophic failure of particles resulting in a number of fragments of substantially smaller sizes than the parent particle . Chipping entails detachment of small fragments from corners and edges , the resulting size spectrum of the progeny is discontinuous in the sense that it comprises one particle of size comparable to the parent particle and a number of fine fragments . For a sharp distinction between the chipping and abrasion mechanisms it is desirable to conceptualize that in the latter phenomenon reduction in particle size occurs by continuous removal of fine matter from its surface due to rubbing and friction . In theory at least , if impact fracture and chipping are the events operating and only a fraction of material is selected for breakage by these mechanisms at any instant of time , then no matter how long the material is ground , some amount will always remain unground . On the other hand when abrasion mechanism also

prevails then, by definition , all the particles in the initial feed must undergo some size reduction even when the grinding time is very short . Neither the continuous variable grinding model in Eq. (2.36) and , therefore , nor its discrete size versions in Eqs. (2.79) and (2.86) take cognizance of the abrasion mechanism which should contribute not insignificantly to the overall size reduction process when relatively soft materials are ground , e.g. limestone .

In our approach we consider a grinding system in which the particulate charge undergoes size reduction by fracture , chipping , as well as , abrasion . It is assumed that the particle population is sufficiently large such that meaningful statistical averages may be defined for the various functions employed in formulation of the grinding model . Furthermore , we shall be concerned with a system which is time-invariant and is also linear in the particle mass fractions $M(x,t)$. We next define a mill function in the frequency form as $\theta(v,x,\underline{\underline{t}})$, $x \leq v$, where $\theta(v,x,\underline{\underline{t}})\,dx$ is the fraction of ground particles in the size range x to x + dx when an unit impulse of size v is ground for a fixed time interval $\underline{\underline{t}}$. Clearly θ is a normalized frequency function , hence ,

$$\int_0^v \theta(v,x,\underline{\underline{t}})\,dx = 1 \qquad (8.1)$$

The mill function in form of a cumulative distribution function can be written as

$$\bar{\theta}(v, x, \bar{\bar{t}}) = \int_0^x \theta(v, w, \bar{\bar{t}})\, dw \qquad (8.2)$$

For a set of batch grinding experiments equally spaced in time , where each time interval is of duration $\bar{\bar{t}}$, let the successive time intervals be denoted by k=1, 2, 3, ... and the grinding time at the end of k-th time interval by t_k . Applying mass balance on the particles of each size , we can directly write the following expressions for the output of the mill after $\bar{\bar{t}}$ grinding time .

$$M_1(t_k) = M_1(t_{k-1}) - \int_{x_1}^{x_0} M(v, t_{k-1})\, \bar{\theta}(v, x_1 , \bar{\bar{t}})\, dv \ , \ i=1 \ ,$$

$$k=1, 2, 3, \ldots \qquad (8.3)$$

and for i > 1 , we have

$$M_i(t_k) = M_i(t_{k-1}) - \int_{x_i}^{x_{i-1}} M(v, t_{k-1})\, \bar{\theta}(v, x_i , \bar{\bar{t}})\, dv$$

$$+ \int_{x_{i-1}}^{x_0} M(v, t_{k-1}) \left[\bar{\theta}(v, x_{i-1} , \bar{\bar{t}}) - \bar{\theta}(v, x_i , \bar{\bar{t}}) \right] dv \qquad (8.4)$$

Substitution of Eq. (2.35a) in Eqs. (8.3) and (8.4) , followed by integration by parts gives

$$M_1(t_k) = 1 - \bar{\theta}(x_o , x_1 , \bar{\bar{t}}) + \int_{x_1}^{x_o} F(v,t_{k-1})\, \theta(v,x_1,\bar{\bar{t}})\, dv \qquad (8.5)$$

and for $i > 1$

$$M_i(t_k) = 1 - F_{i-1}(t_{k-1}) + \bar{\theta}(x_o , x_{i-1} , \bar{\bar{t}}) - \bar{\theta}(x_o , x_i , \bar{\bar{t}}) + \int_{x_i}^{x_o} F(v,t_{k-1})\, \theta(v,x_i , \bar{\bar{t}})\, dv - \int_{x_{i-1}}^{x_o} F(v,t_{k-1})\, \theta(v,x_{i-1} , \bar{\bar{t}})\, dv \qquad (8.6)$$

Since $M(v,t_k)$ is not directly accessible from size distribution data , the advantage of Eqs. (8.5) and (8.6) over Eqs. (8.3) and (8.4) lies in the fact that it is convenient to fit suitable orthogonal polynomials to $F(x_i , t_k)$ in x . The resulting expression is then substituted in Eqs. (8.5) and (8.6) for analytical evaluation of the integral terms .

It is obvious that the knowledge of the function $\bar{\theta}$ is sufficient for identification of the system model , which is now discrete in time . The function $\bar{\theta}$ is subject to a number of constraints :

(i) $\bar{\theta}(v, 0, \bar{\bar{t}}) = 0$, for all v

(ii) $\bar{\theta}(v,v,\bar{\bar{t}}) = 1$, for all v

(iii) $\bar{\theta}(v,x,\bar{\bar{t}})$ is a monotonically non-decreasing function in x , i.e. , $\theta(v,x,\bar{\bar{t}}) \geq 0$, for $x \leq v$.

8.2 Determination of the Mill Function and Demonstration of the Proposed Identification Scheme

For a demonstration of the proposed method of identification , we have chosen Berlioz grinding data | 108 | for dolomite ground dry in a batch ball mill with particle loads of 3300 gm. and 2640 gm. . In absence of any 'a priori' knowledge of the functional form of $\bar{\theta}$ we assumed the following generalized polynomial expression

$$\bar{\theta}(v,x,\bar{\bar{t}}) = a_0 + a_1 x + a_2 v + a_3 x^2 + a_4 xv + a_5 v^2 + a_6 x^3 + a_7 x^2 v + a_8 x v^2 + a_9 v^3 + \ldots \qquad (8.7)$$

It can easily be shown that under the first two constraints mentioned above Eq. (8.7) reduces to

$$\bar{\theta}(v,x,\bar{\bar{t}}) = \left(\frac{x}{v}\right) + x(v-x)\left[b_0 + b_1 x + b_2 v + b_3 x^2 + b_4 xv + b_5 v^2 + \ldots \right] \qquad (8.8)$$

where

$$b_0 = - a_3$$

$$b_1 = - a_6$$

$$b_2 = - a_7$$

..... and so on .

Without violating the first two constraints , Eq.(8.8) can be modified to perhaps a more versatile form

$$\bar{\bar{\theta}}(v,x,t) = \left(\frac{x}{v}\right)^{\beta_1} + x^{\beta_2} v^{\beta_3} (v^{\beta_4} - x^{\beta_4}) \times [b_0 + b_1 x + b_2 v + b_3 x^2 + b_4 xv + b_5 v^2 + \ldots] \qquad (8.9)$$

where the exponents β_1 , β_2 and β_4 are greater than zero . It is noted that Eq. (8.8) is a special case of Eq. (8.9) when $\beta_1 = \beta_2 = \beta_4 = 1$ and $\beta_3 = 0$. The coefficients b's and β_1 , β_2 , β_3 and β_4 were determined from experimental data $\hat{M}_j(t_k)$ by minimizing a least squares type error function

$$Er = \sum_{k=1}^{3} \sum_{i=1}^{12} [\bar{M}_i(t_k) - \hat{M}_i(t_k)]^2 \qquad (8.10)$$

Data for three grinding time intervals t_k = 60 , 80 , and 100 revolutions was employed ($\bar{t}$ = 20 mill rev.) . A combination of random search method | 109-115 | for initial search followed by Rosenbrock method | 116-119 | for fine convergence was used for the non-linear optimization . Initial trials showed that in the polynomial in Eq. (8.9) the constant b_0 ,

the coefficients of pure x terms , the cross product terms , and the terms of order more than 5 were about 3 orders of magnitude smaller than the coefficients of first five pure v terms . Elimination of these terms in the polynomial did not increase the value of the error function Er by more than 2 per-cent . Consequently , the form of the function $\bar{\theta}$ was modified to

$$\bar{\bar{\theta}}(v,x,\bar{\bar{t}}) = \left(\frac{x}{v}\right)^{\beta_1} + x^{\beta_2} v^{\beta_3} \left(v^{\beta_4} - x^{\beta_4}\right) \times \left[c_1 v + c_2 v^2 + c_3 v^3 + c_4 v^4 + c_5 v^5\right] \qquad (8.11)$$

which had only 9 parameters to be estimated . The values of these parameters , and the experimental and computed size distributions for 3300 gm. load of particulate charge are shown in Tables 1 and 2 , respectively . It can be seen that the predictions of the size distributions are very good . Indeed , in comparison with the large number of simulation attempts reported in the literature , this seems to be the closest that has ever been achieved .

In order to reduce the number of parameters in function $\bar{\theta}$, a 7 parameter form was next tried

$$\bar{\bar{\theta}}(v,x,\bar{\bar{t}}) = \left(\frac{x}{v}\right)^{\bar{\beta}_1} + \left[\left(\frac{x}{v}\right)^{\bar{\beta}_2} - \left(\frac{x}{v}\right)^{\bar{\beta}_1}\right] \times \left[d_1 v + d_2 v + \ldots + d_5 v^5\right] \qquad (8.12)$$

TABLE - 8.1

Estimated values of the parameters in Eq.(8.11) for 3300 gm. of particle load.

Parameter	Value
β_1	43.827177
β_2	00.890110
β_3	-43.721428
β_4	42.835910
c_1	0.482328
c_2	-1.660828
c_3	2.179485
c_4	-0.373028
c_5	-0.378885

TABLE - 8.2

Comparison of Experimental and Computed Size Distributions, Berlioz Data for 3300 gm. Load of Dolomite |108|

Size index i	40 rev.	60 rev.		80 rev.		100 rev.	
	Exptl.	Exptl.	Predicted	Exptl.	Predicted	Exptl.	Predicted
1	62.16	49.64	49.54	38.88	39.48	31.03	31.46
2	13.93	16.22	16.37	17.25	17.06	17.06	17.06
3	8.28	10.76	10.48	12.23	12.22	13.29	13.51
4	4.29	6.10	6.32	8.22	8.10	9.65	9.62
5	2.80	4.12	4.30	5.66	5.66	6.80	6.89
6	1.92	3.04	3.04	3.98	4.09	4.87	5.05
7	1.60	2.42	2.42	3.17	3.20	4.03	3.95
8	1.13	1.73	1.77	2.21	2.38	2.98	2.98
9	0.69	1.14	1.18	1.75	1.67	2.06	2.13
10	0.64	0.93	0.98	1.33	1.33	1.54	1.68
11	0.50	0.75	0.76	0.97	1.03	1.27	1.30
12	0.44	0.69	0.64	0.93	0.84	1.14	1.04

The estimated values of the parameters for 3300 gm. as well as 2640 gm. load of particulate charge are given in Table 3 . The predicted size distributions for 3300 gm. load were now within $\pm$ 1 per-cent of those reported in Table 2 . The experimental and computed size distributions for 2640 gm. load are shown in Table 4 . Again, the agreement is quite good .

For both the sets of data , four to five widely different starting guesses for the parameters in the three polynomials in Eqs. (8.10) , (8.11) and (8.12) were tried . The final parameter estimates did not show more than $\pm$ 1 per-cent variation . The variation in the value of error function was also within $\pm$ 2 per-cent . Further evidences in support of the validity and the accuracy of the estimated mill function shall be given in the next chapter .

Figures 8.1 and 8.2 show the nature of $\bar{\theta}(v,x,\bar{t})$ curves plotted as a function of dimensionless size x/x_i on abscissa scale , for feed impulses of size v located at various mesh openings $v = x_i$. In most favourable case when the impulse is located at 10 mesh opening (v=1680 microns) about 18 per-cent material only is ground finer than the adjacent 14 mesh (x=1190 microns) in 20 mill revolutions . Indeed, more than 75 per-cent product lies in the size range

TABLE - 8.3

Estimated values of the parameters in Eq. (8.12) for 3300 gm. and 2640 gm. particle loads .

Parameter	Numerical Value of the Parameter	
	3300 gm. particle load	2640 gm. particle load
$\bar{\beta}_1$	43.732543	37.279627
$\bar{\beta}_2$	0.907984	0.827502
d_1	0.482858	0.469503
d_2	-1.660334	-1.587290
d_3	2.179932	2.202055
d_4	-0.372955	-0.385656
d_5	-0.378812	-0.412245

TABLE - 8.4

Comparison of Experimental and Computed Size Distributions, Berlioz Data for 2640 gm. Load of Dolomite | 108 |

Size index i	40 rev. Exptl.	60 rev. Exptl.	60 rev. Predicted	80 rev. Exptl.	80 rev. Predicted	100 rev. Exptl.	100 rev. Predicted
1	60.05	46.88	46.90	36.56	36.63	28.11	28.61
2	13.59	15.43	15.40	15.82	15.80	15.41	15.33
3	8.54	11.01	10.68	12.71	12.26	13.65	13.35
4	4.87	6.73	6.94	8.21	8.69	9.42	10.13
5	3.13	4.66	4.75	6.06	6.18	7.35	7.41
6	2.23	3.38	3.48	4.42	4.60	5.47	5.59
7	1.82	2.84	2.78	3.84	3.65	4.85	4.44
8	1.40	2.09	2.15	2.81	2.84	3.55	3.48
9	0.94	1.40	1.52	1.87	2.08	2.26	2.60
10	0.74	1.09	1.19	1.48	1.62	2.01	2.02
11	0.56	0.79	0.90	1.10	1.24	1.42	1.57
12	0.54	0.94	0.86	1.14	1.11	1.52	1.36

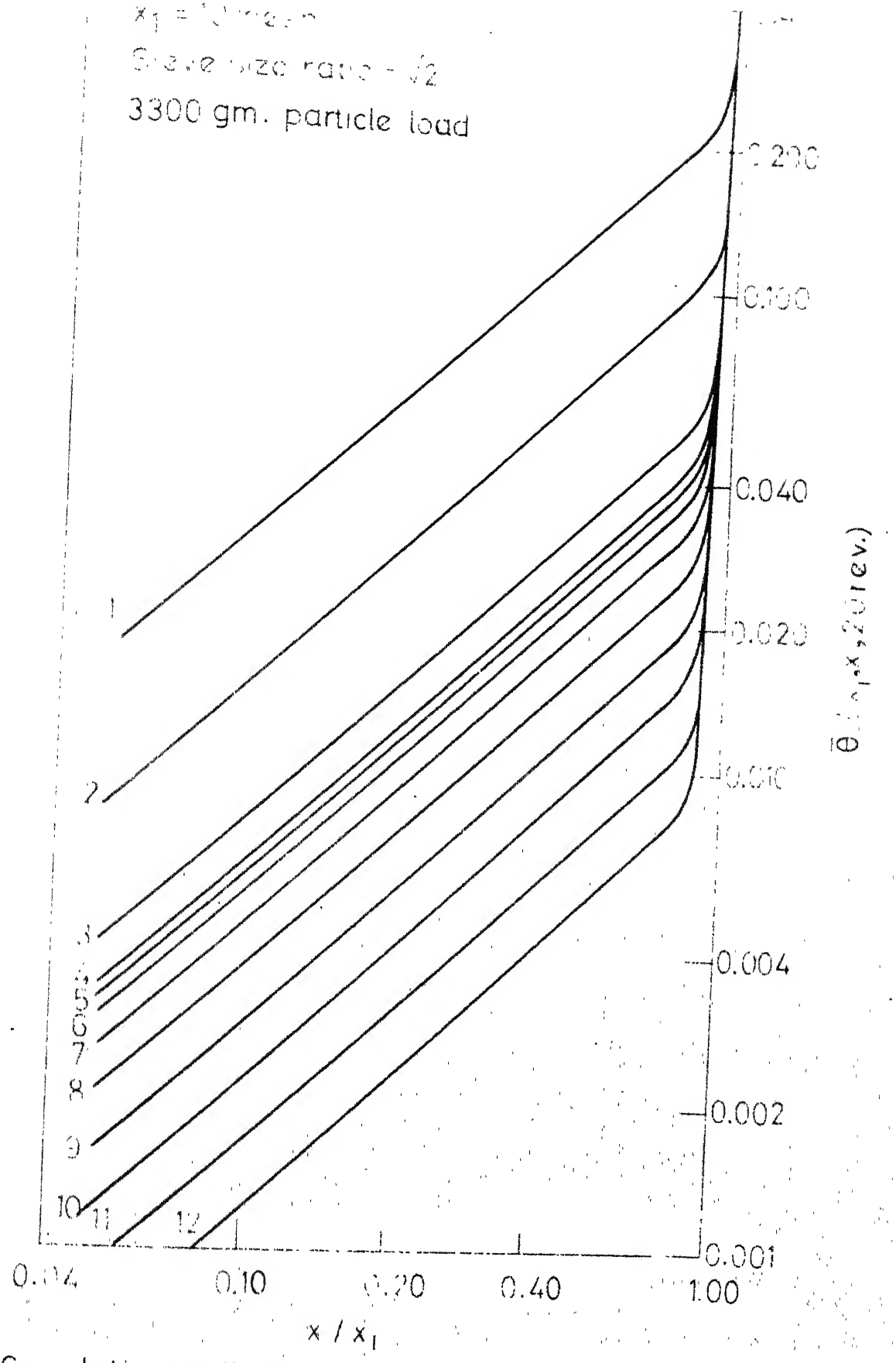


Fig.8.1 Cumulative mill function as a function of scaled size x/x_i for feed impulses of size x_i located at various mesh openings

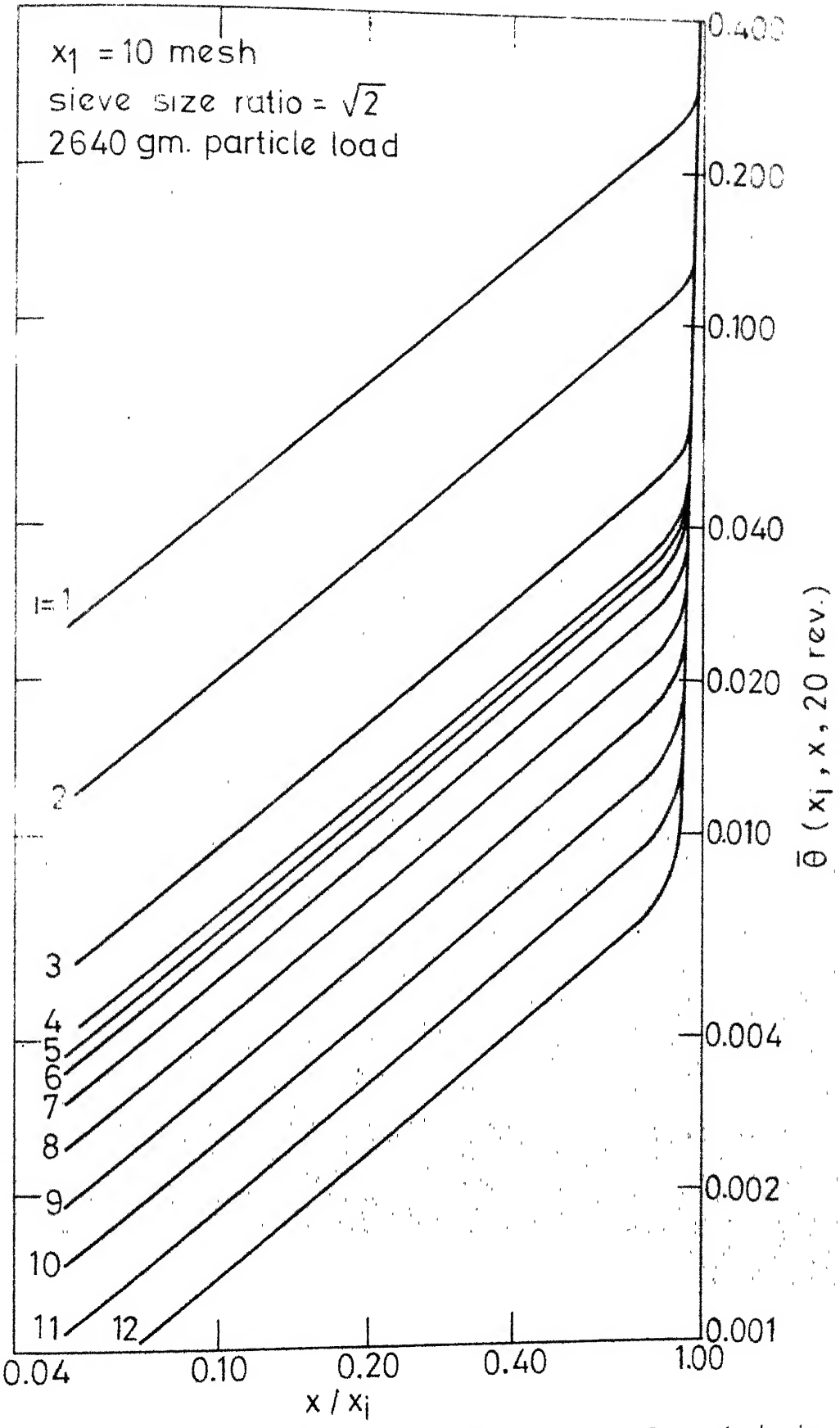


Fig. 8.2 Cumulative mill function as a function of scaled size x/x_i for feed impulses of size v located at various mesh

x_i to 0.9 x_i . This strongly indicates that in course of the size reduction , chipping and abrasion played a significant role . The effect is even more pronounced when the feed impulse is located at smaller sizes .

It is interesting to note that in our choice of the function $\bar{\theta}$ in Eqs. (8.11) and (8.12) we could not guarantee the third constraint regarding the monotonic nondecreasing nature of this function . However , the computed curves show that this condition is not violated .

Figures (8.3) and (8.4) show $\bar{\theta}(v, x_i , \bar{\bar{t}})$ curves , that is, the fraction of feed material ground out of a sieve size interval , when the feed impulse ranges over two adjacent mesh openings , $x_i \leq v \leq x_{i-1}$. In all cases , when the impulse is located anywhere in the size range $[x_i + 0.2 (x_{i-1} - x_i)]$ to x_{i-1} , most of the material after 20 mill revolutions is still coarser in size than the lower mesh opening x_i . In other words , a large proportion of solids after size reduction still remains within the original size interval . This result confirms that the inner breakage parameter $b_{i,i}$ is not negligible , as was pointed out earlier in chapter 6 also .

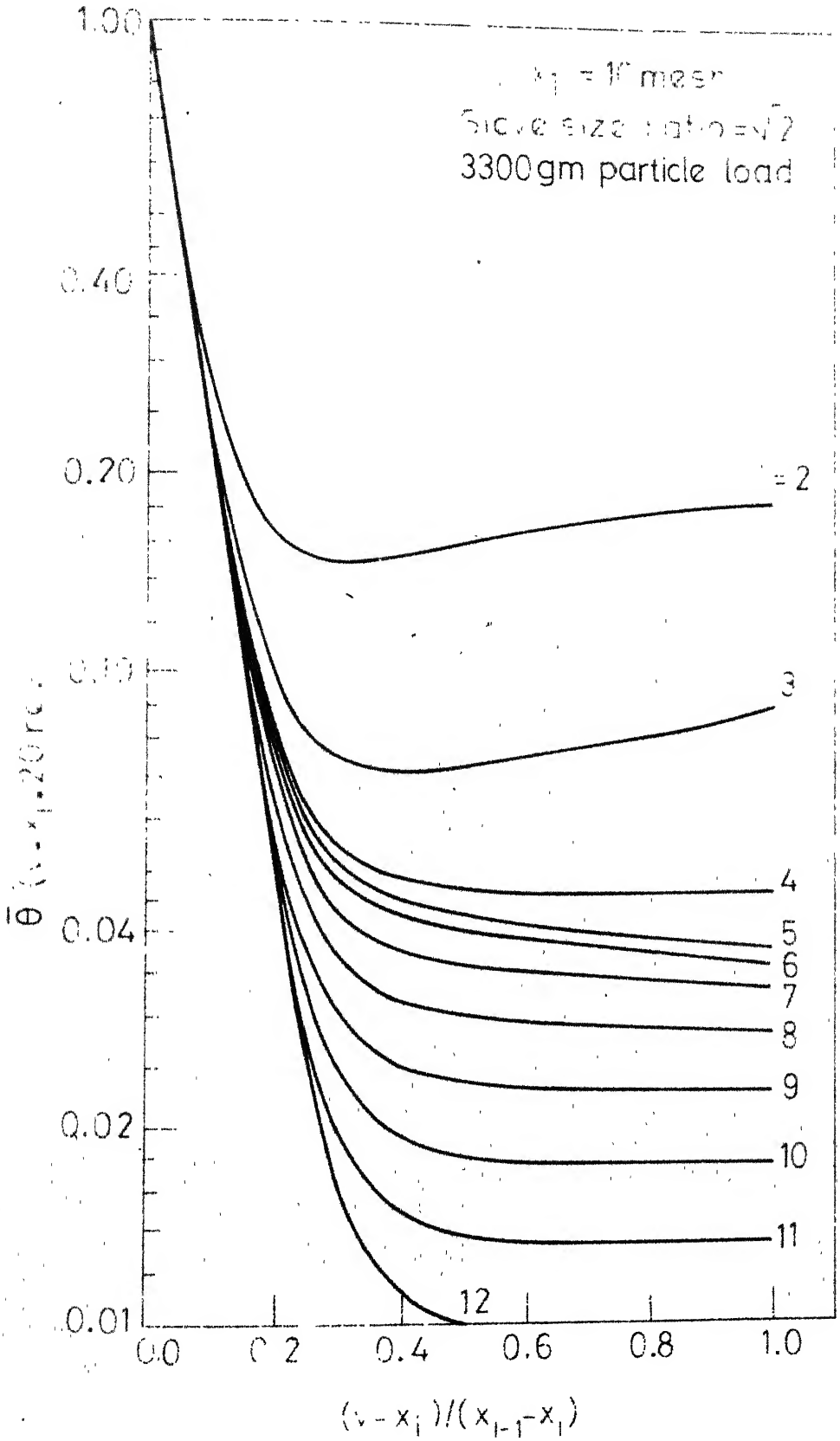


Fig 5.3 Cumulative mill function for $x = x_i$ as a function of $(v-x_i)/(x_{i-1}-x_i)$, representing the fraction of feed material ground out of sieve size interval i when the feed impulse ranges over the size range $x_i < v < x_{i-1}$. 3300gm particle load.

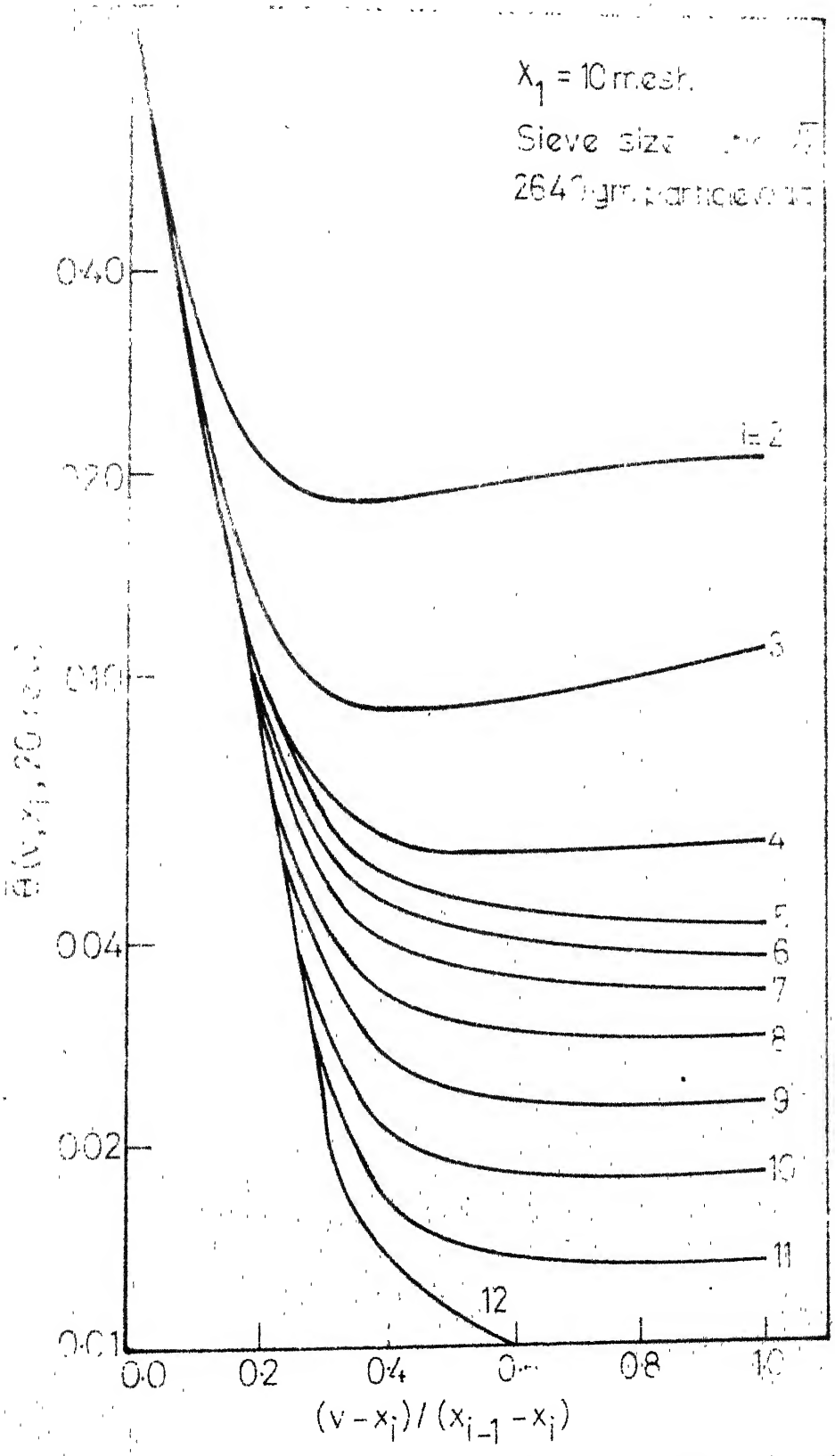


Fig. 8·4 – Cumulative mill function for $x = x_i$ as a function of $(v-x_i)/(x_{i-1}-x_i)$, representing the fraction of feed material ground out of sieve size interval i when the [illegible] passes over the size range $x_i < v \leq x_{i-1}$

8.3 Conclusions

A batch ball mill grinding system has been identified in terms of a mill function $\underline{\underline{\Theta}}(v,x,t)$ using a continuous size and discrete time mathematical model . The procedure presented here seems to be of reasonable generality . The role of chipping and abrasion mechanisms has also been properly delineated .

The proposed identification scheme has the following advantages :-

(i) The mill function provides more insight into the breakage behaviour of the particles than that can be obtained from the size discretized parameters .

(ii) In contrast to the size discretized parameters , which do not permit a unique identification of the grinding system because of their dependence on the particle spectra , mill function provides a more sound and uniform basis for comparison of different grinding systems .

(iii) The task of parameter estimation is considerably simplified when the mill function is used for identification. For example , in case of the two grinding systems considered above , only 7-9 unknown constants were required to be estimated . Whereas , for 12

discrete size intervals, 78 time dependent size discretized parameters must be estimated .

(iv) The proposed identification scheme is economical from the point of view of experimental labour , too. Only two to four batch grinding experiments , equally spaced in time , are sufficient for the identification of the grinding system .

(v) The discrete time mill function can be empolyed for estimation of S_i and $B_{i,j}$ parameters in the more commonly used continuous time and discrete size grinding model 'A' in Eq. (2.79) , as shown in the next chapter .

CHAPTER 9

ESTIMATION OF RATE AND BREAKAGE PARAMETERS

In absence of any theoretical basis for prediction of $n(n+1)/2$ time varying rate and breakage parameters in Eq. (5.10), the parameter estimation from noise corrupted experimental data for n as large as 10 or more, is not an easy task. During the course of present investigation, a number of methods based on quasi-linearization [120, 121], Kalman filter [122, 123] and also parameter optimization using polynomial approximation for time variation were tried, which resulted in little success. It is then not surprising that all previous work in the area of parameter estimation has been based on simplified models of the grinding kinetics, as reviewed earlier in Chapter 2. In this chapter, our objective is to estimate these parameters using minimum possible simplifications. Since, we have had little success in tracking the grinding parameters in time, in order to make the problem tractable we have assumed that for appropriate batch grinding data obtained over short grinding time intervals of duration $\bar{t}$, the grinding parameters are sensibly constant in any given time interval k in the time

range $t_{i-1} \leq t \leq t_k$. Hence the grinding model becomes

$$\frac{dM_i(t)}{dt} = - S_i(k)\, M_i(t) + \sum_{j=1}^{i-1} S_j(k)\, B_{i,j}(k)\, M_j(t) ,$$

$$i = 1,2, \ldots \quad n,$$

$$k = 1,2, \ldots \quad , \qquad (9.1)$$

$$t_{k-1} \leq t \leq t_k$$

The estimation procedure presented here is based on mill function θ , which has been defined and estimated in the previous Chapter .

9.1 Development of the Proposed Scheme

From Eq. (2.103) it follows that solution to Eq. (9.1) is

$$\underline{M}(t_k) = \underline{\underline{O}}(k)\ \underline{M}(t_{k-1}) \qquad (9.2)$$

where $\underline{M}(t_k)$ is the n x 1 vector of the product size distribution, $\underline{M}(t_{k-1})$ is the feed size distribution vector at the start of the k-th grinding interval , $\underline{\underline{O}}(k)$ is a n x n lower triangular matrix operator which maps the feed vector to product vector for k-th time interval . The element $o_{i,j}(k)$ of the matrix $\underline{\underline{O}}$ is the

fraction of feed in j-th size interval that is found in i-th size interval in the product after grinding for $\bar{\bar{t}}$ time duration . It can be seen that the j-th vector in $\underline{\underline{O}}$ matrix can be directly obtained by grinding a single size feed of size j for a time period of duration $\bar{\bar{t}}$. An alternative method of calculating $\underline{\underline{O}}$ matrix, which is based on the mill function, is given below .

By definition of $\bar{\theta}(v,x,\bar{\bar{t}})$, we have

$$o_{i,i}(k) = 1 - \frac{\int_{x_i}^{x_{i-1}} M(v,t_{k-1})\ \bar{\theta}(v,x_i,\bar{\bar{t}})\ dv}{\int_{x_i}^{x_{i-1}} M(v,t_{k-1})\ dv} \qquad (9.3)$$

and

$$o_{i,j}(k) = \frac{\int_{x_j}^{x_{j-1}} M(v,t_{k-1})\ [\bar{\theta}(v,x_{i-1},\bar{\bar{t}}) - \bar{\theta}(v,x_i,\bar{\bar{t}})]\ dv}{\int_{x_j}^{x_{j-1}} M(v,t_{k-1})\ dv} \qquad (9.4)$$

Substituting for $M(v,t_{k-1})$ from Eq. (2.35a), Eqs.(9.3) and (9.4) become

$$o_{i,i}(k) = \frac{\left[F_{i-1}(t_{k-1})\ [\,1 - \bar{\theta}\,(x_{i-1}\ ,\ x_i;\ \bar{\bar{t}})\,] + \int_{x_i}^{x_{i-1}} F(v,t_{k-1})\ \theta\ (v,x_i\ ,\ \bar{\bar{t}})\ dv\right]}{F_{i-1}\,(t_{k-1}) - F_i(t_{k-1})} \qquad (9.5)$$

$$o_{i,j}(k) = \frac{1}{F_{j-1}(t_{k-1}) - F_j(t_{k-1})}$$

$$\times \Big[F_{j-1}(t_{k-1}) \Big[\bar{\theta}\,(x_{j-1}, x_{i-1}, \bar{\bar{t}}\,) - \bar{\theta}(x_{j-1}, x_i, \bar{\bar{t}}) \Big]$$

$$- F_j(t_{k-1}) \Big[\bar{\theta}(x_j, x_{i-1}, \bar{\bar{t}}) - \bar{\theta}(x_j, x_i, \bar{\bar{t}}) \Big]$$

$$- \int_{x_j}^{x_{j-1}} F(v, t_{k-1}) \Big[\theta(v, x_{i-1}, \bar{\bar{t}}\,) - \theta(v, x_i, \bar{\bar{t}}) \Big] dv \Big] \quad (9.6)$$

Substituting the expressions for $\bar{\theta}$ and $F_i(k)$ in Eqs.(9.5) and (9.6) , $\underline{\underline{O}}(k)$ matrices can be evaluated for each time interval k .

Now, if the explicit aljeabric expressions can be obtained for each element $o_{i,j}(k)$ in terms of the paramoters $S_i(k)$ and $B_{i,j}(k)$, it shouldbe possible to back-calculate the parameters one by one . Neither Reid's solution in Eq. (2.95) nor Herbst and Mika's solution in Eq. (2.102) is convenient for this purpose . This difficulty has been overcome by devising a '0,1' variable suppression technique. The procedure is as follows .

It can easily be seen that the diagonal elements

$o_{i,i}$ are simply exp $(- S_i \underline{\underline{t}})$. Hence, all the n rate paramcters can be first obtained immediately .

A close analysis of the structure of the $\underline{\underline{O}}$ matrix reveals that if in a particular row one moves towards left, only one new breakage parameter is encountered in every column . Moreover, the basic structure of the element $o_{i,j}$ is as follows

$$o_{i,j} = \pi_{i,j} + B_{i,j}\, \epsilon_{i,j} \qquad (9.7)$$

It will be further seen that if all the parameters are known, substitution of 1 for M_j and 0 for the rest of the M's in the Reid's solution to $M_i(t)$ in Eq. (2.95) would immediately give us the value which will be equal to $o_{i,j}$. Since, we do not know $B_{i,j}$, we first calculate $o_{i,j}$ as described above assigning $B_{i,j}$ a value 0 . This gives us the constant $\pi_{i,j}$ in Eq. (9.7) . Repeating the same calculation again, and now assigning $B_{i,j}$ a value 1 , we get a numerical quantity which is equal to $(\pi_{i,j} + \epsilon_{i,j})$. Hence, using the known actual value of $o_{i,j}$ obtained from the mill function, and Eq. (9.7) , the parameter $B_{i,j}$ can be calculated . It follows that

$$B_{i,j} = \frac{o_{i,j} - \pi_{i,j}}{\epsilon_{i,j}} \qquad (9.8)$$

In this way all the breakage parameters can be determined sequentially. It should be pointed out that unlike Austin and Luckie's approximate back calculation method [78] , the back calculation procedure described above is exact and, therefore, no formal verification is as such necessary . It may, however, be mentioned that when tested against synthetic data, the above method did provide the exact estimates of the parameters , as expected . For clear illustration of the procedure, a flow chart has been given in Appendix 3 .

9.2 Estimation of Time Varying Parameters for Two Real Grinding Systems

Since the estimates of the mill functions $\bar{\theta}$ and the orthogonal polynomials for $F_i(k)$ are already known for two of the batch grinding systems studied by Berlioz [108] , these may be substituted in Eqs. (9.5) and (9.6) in order to evaluate $\underline{\underline{O}}(k)$ matrices for grinding time intervals 20-40 , 40-60, 60-80, 80-100 and 100-120 mill revolutions . For each time interval the parameters $S_i(k)$ and $B_{i,j}(k)$ can be obtained using the exact back calculation method described above.

Figures 9.1 and 9.2 show the variations in the rate parameters with time for 3300 gm. and 2640 gm. loads of dolomite . Figures 9.3 and 9.4 show plots of some of the breakage parameters as a function of time . We note that many parameters , especially the parameters pertaining to coarse size intervals, undergo a distinct change from first to fifth time interval . In general, the time dependence is more pronounced for the rate parameters than for the breakage parameters . The breakage parameters are not difference similar as widely assumed hitherto | 42, 58, 61, 64, 72 | . The breakage parameter $B_{i+1, i}$ is considerably on the higher side when the size index i is greater than 5 or so. This again reflects the predominance of chipping and abrasion mechanisms in the size reduction of fine particles .

9.3 Definition and Calculation of the Reference Parameters

Since model parameters are function of particle size distribution within the discrete size intervals, it is essential to fix a standard reference for comparison with different grinding systems . The most convenient choice is a feed of uniformly distributed size distribution,

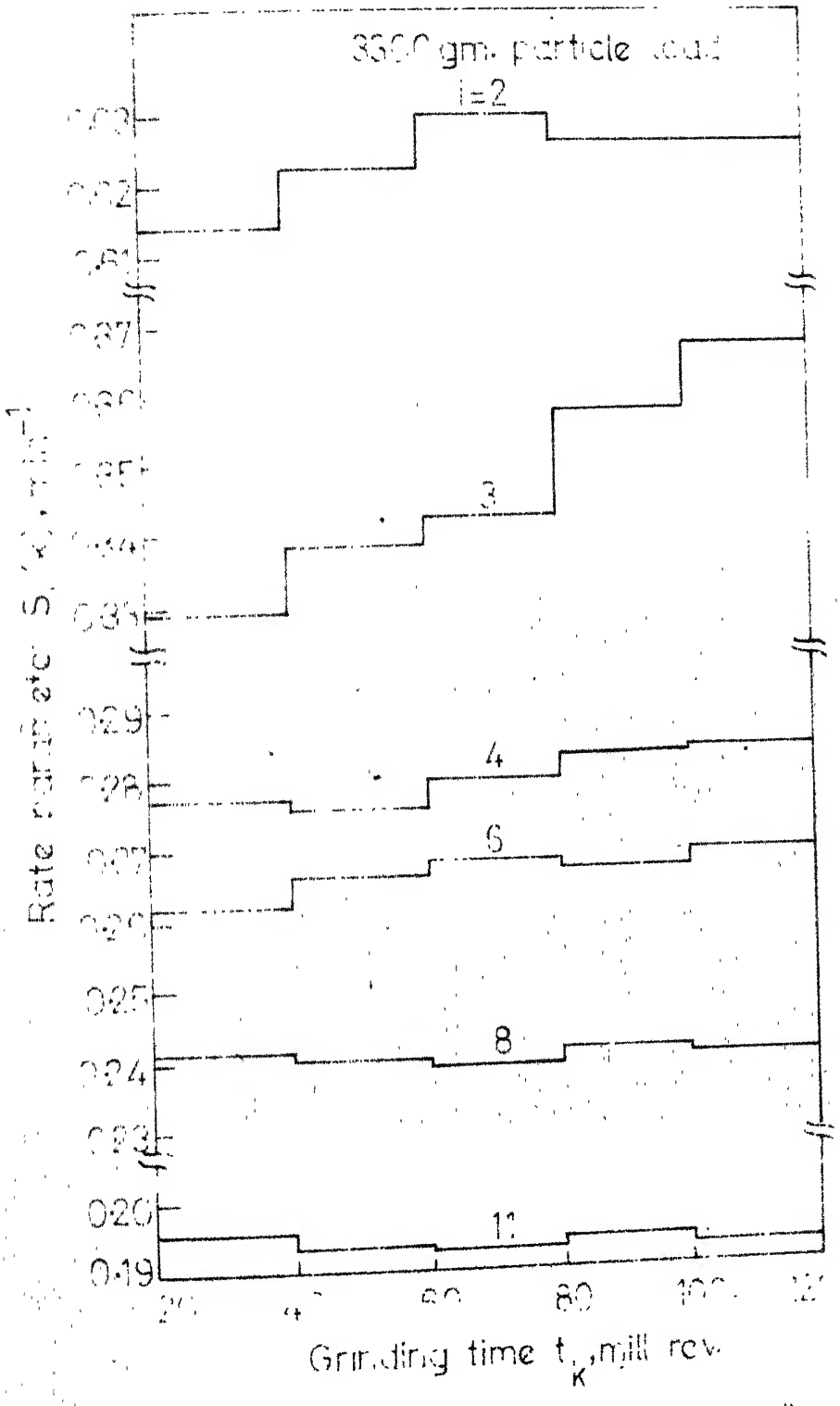


Fig. 9.1 Variation of rate parameters over discrete intervals of time for 3300 gm. particle load.

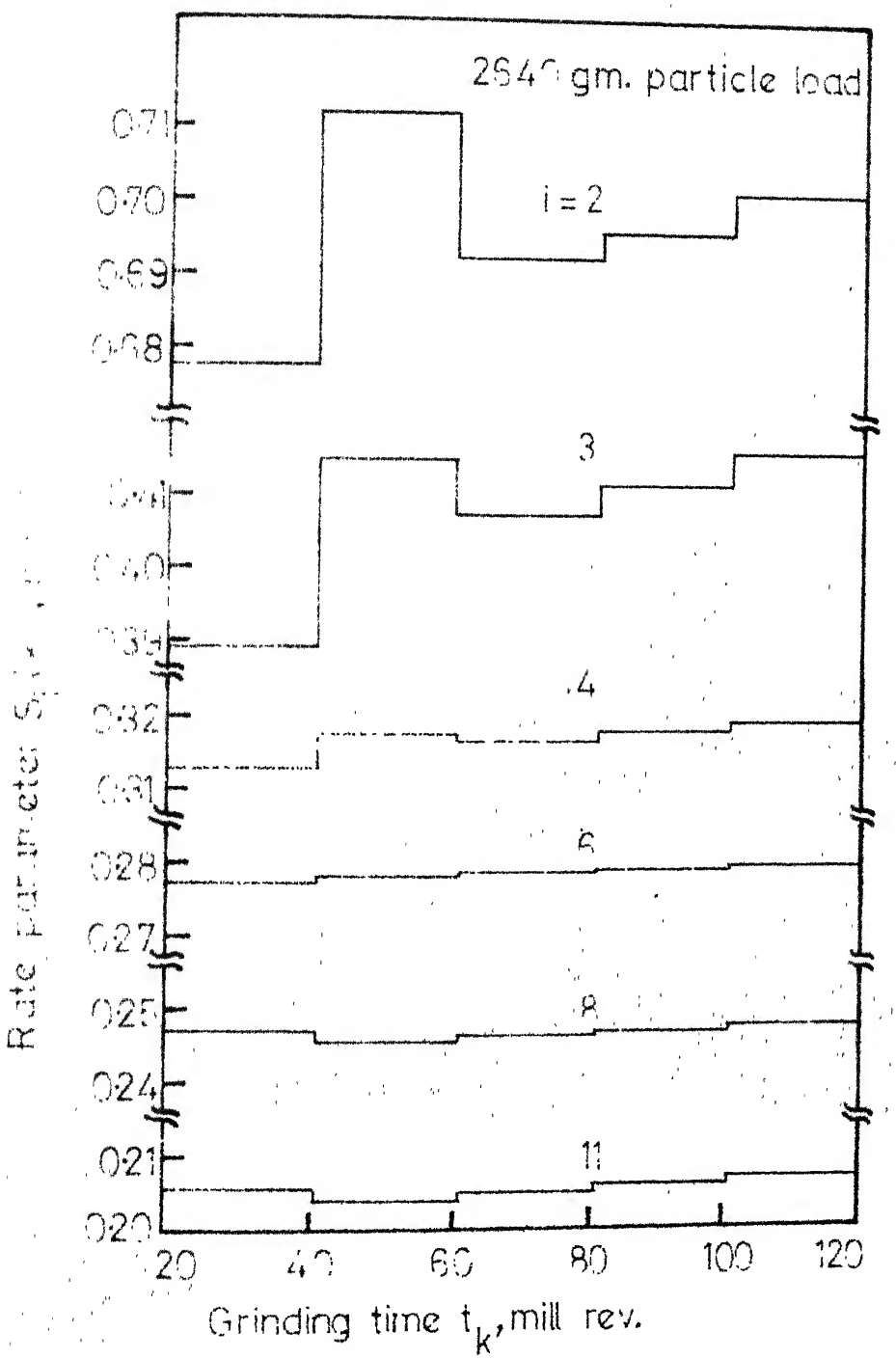


Fig. 9·2 Variation of rate parameters over discrete intervals of time for 2640 gm. particle load

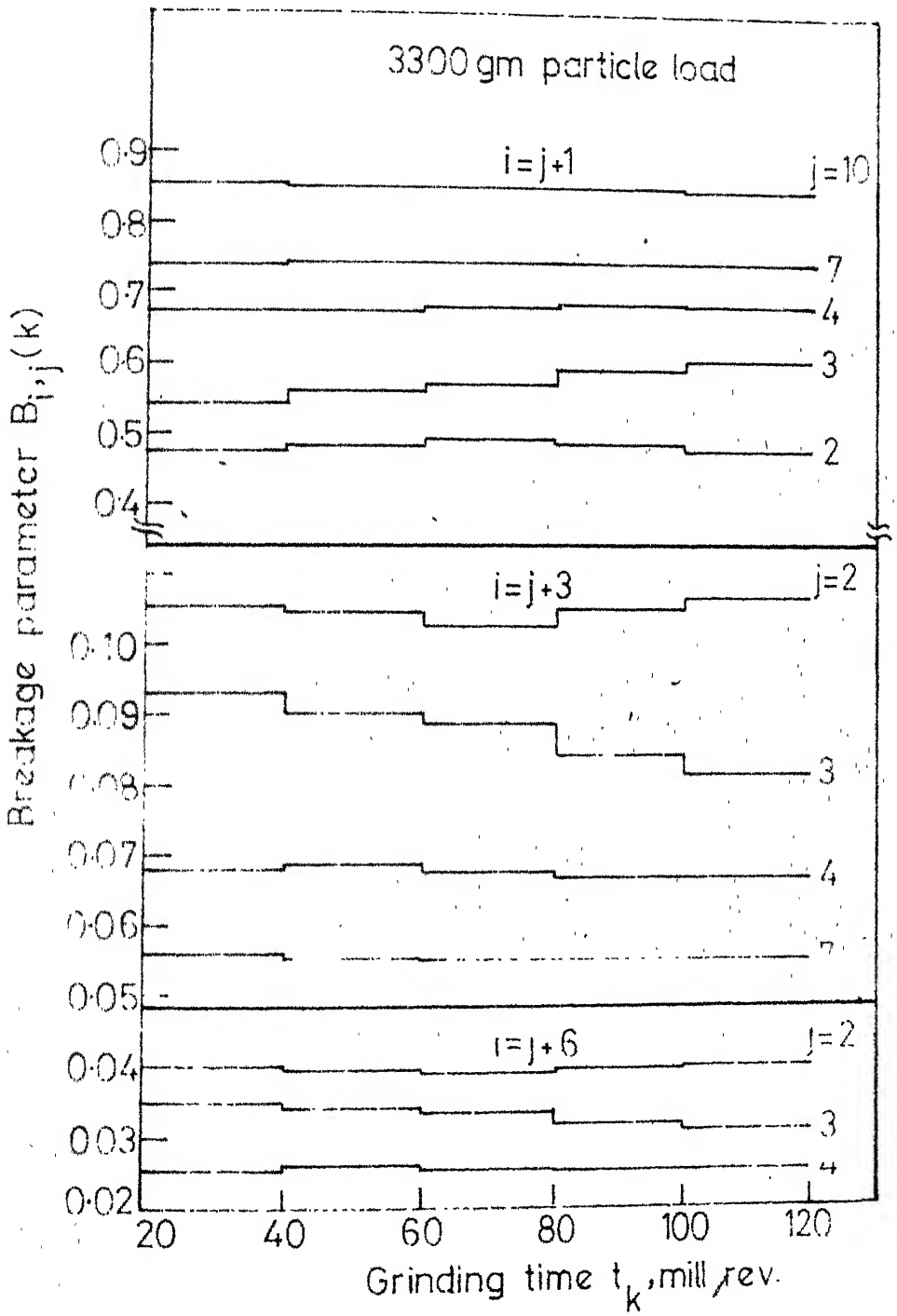


Fig. 9·3 – Variation of breakage parameters over discrete intervals of time for 3300 gm. particle load.

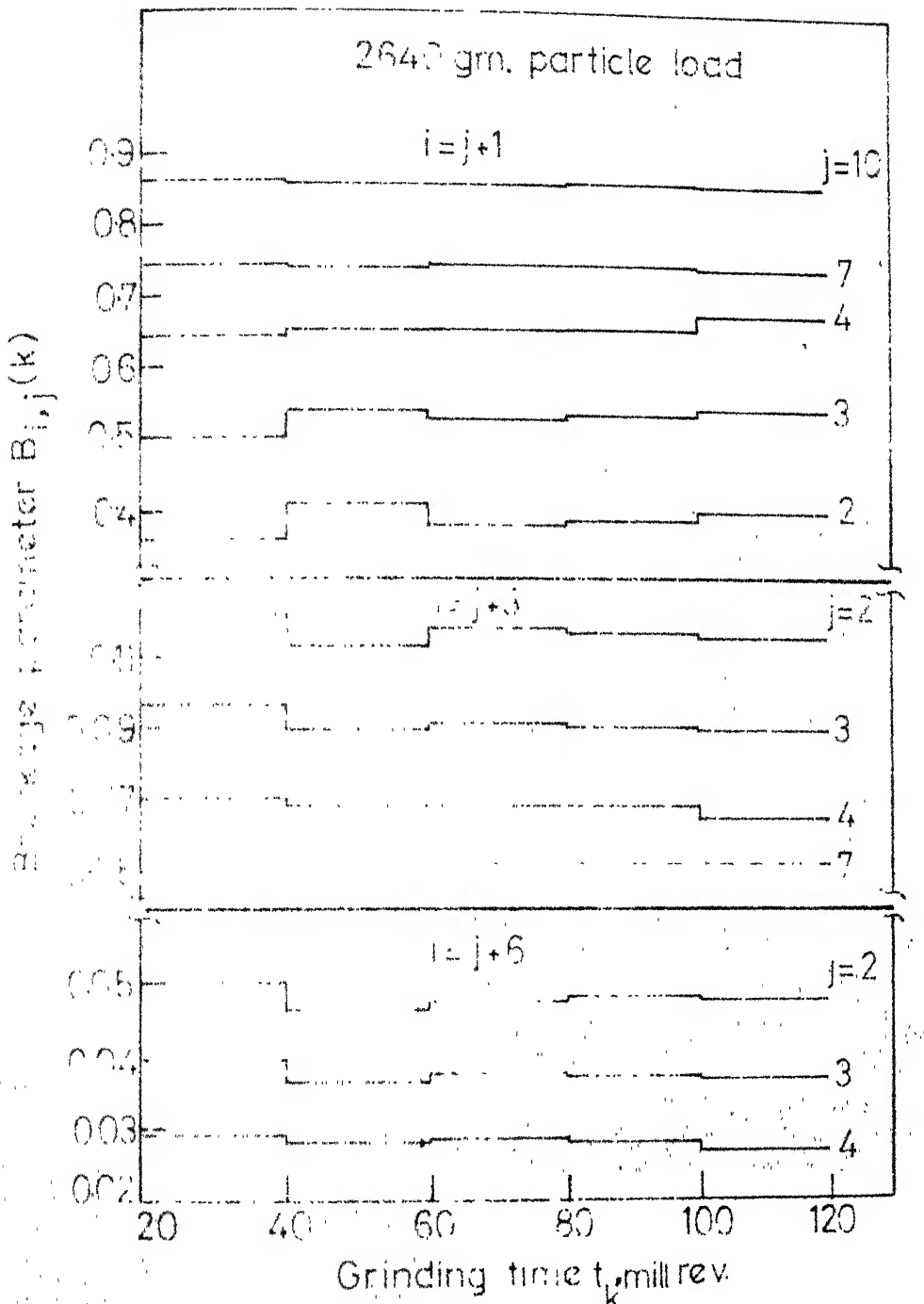


Fig. 9.4 Variation of breakage parameters over discrete intervals of time for 2640 gm. particle load

ground for a fixed period of time $\bar{\bar{t}}$. For a uniformly distributed feed Eqs. (9.3) and (9.4) reduce to

$$o_{i,i} = 1 - \frac{\int_{x_i}^{x_{i-1}} \bar{\theta}(v, x_i, \bar{\bar{t}})\, dv}{x_{i-1} - x_i} \qquad (9.9)$$

$$o_{i,j} = \frac{\int_{x_j}^{x_{j-1}} \left[\bar{\theta}(v, x_{i-1}, \bar{\bar{t}}) - \bar{\theta}(v, x_i, \bar{\bar{t}}) \right] dv}{x_{j-1} - x_j} \qquad (9.10)$$

Thus , applying the back calculation method described above to the $\bar{\bar{O}}$ matrix whose elements are given in Eqs. (9.9) and (9.10) , both sets of the reference parameters can be obtained .

The lower triangular parameter matrices in Tables 9.1 and 9.2 give the values of the reference parameters for two of the Berlioz's grinding systems. The diagonal elements are the rate parameters for respective size intervals i. The off - diagonal elements in i-th row and j-th column are breakage parameters $B_{i,j}$.

Mika et al. | 42 | have reported the values of the feed size rate parameters , estimated from log-linear disappearance kinetics plots, for these two grinding

TABLE - 9.1

ference Parameters for Berlioz Grinding System. 3300 gm. Load of Dolomite |108| , $\bar{t}$ = 20 mill rev.

agonal Elements are Rate Parameters and Off-Diagonal Elements are Breakage Parameters

j	1	2	3	4	5	6	7	8	9	10	11	12
.	0.6128											
2	0.4735	0.6228										
3	0.1420	0.4764	0.3654									
4	0.1065	0.1387	0.6063	0.2876								
5	0.0767	0.1065	0.0985	0.6916	0.2740							
6	0.0555	0.0767	0.0818	0.0724	0.7109	0.2672						
7	0.0401	0.0555	0.0586	0.0654	0.0663	0.7214	0.2546					
8	0.0290	0.0401	0.0424	0.0468	0.0616	0.0628	0.7427	0.2380				
9	0.0209	0.0290	0.0307	0.0339	0.0440	0.0595	0.0562	0.7746	0.2210			
10	0.0152	0.0210	0.0223	0.0246	0.0319	0.0425	0.0554	0.0463	0.8123	0.2060		
11	0.0110	0.0152	0.0161	0.0178	0.0231	0.0309	0.0396	0.0493	0.0347	0.8512	0.1934	
12	0.0080	0.0111	0.0118	0.0130	0.0169	0.0225	0.0288	0.0352	0.0421	0.0229	0.8879	0.1838

TABLE - 9.2

Reference Parameters for Berlioz Grinding System, 2640 gm. Load of Dolomite | 108 |, $\bar{t}$ = 20 mill
Diagonal Elements are rate parameters and off-diagonal Elements are Breakage Perameters .

1	2	3	4	5	6	7	8	9	10	11	12
0.6673											
0.4735	0.7528										
0.1303	0.1600	0.4464									
0.1019	0.1311	0.5374	0.3393								
0.0751	0.1046	0.0932	0.6870	0.3121							
0.0558	0.0775	0.0819	0.0641	0.7230	0.2993						
0.0415	0.0576	0.0603	0.0639	0.0535	0.7421	0.2843					
0.0308	0.0428	0.0448	0.0468	0.0574	0.0477	0.7671	0.2668				
0.0230	0.0319	0.0334	0.0349	0.0419	0.0539	0.0403	0.8000	0.2496			
0.0171	0.0238	0.0249	0.0260	0.0313	0.0393	0.0493	0.0306	0.8373	0.2344		
0.0128	0.0177	0.0186	0.0194	0.0234	0.0294	0.0359	0.0434	0.0198	0.8745	0.2221	
0.0096	0.0133	0.0139	0.0145	0.0175	0.0220	0.0269	0.0315	0.0368	0.0091	0.9087	0.2124

systems. They have also reported the values of the parameters $B_{i,1}$, which were determined from estimated values of constants γ_i and Eq. (2.109) . A comparison of these parameter estimates with the reference parameters determined above, is presented in Table 9.3 . It can be seen that except for $B_{2,1}$ and $B_{3,1}$, in both cases the agreement is quite satisfactory. It should be noted that in case of first two breakage parameters , Mika et al. have indicated that there existed inherent uncertainties in the procedure adopted by them. Moreover , it should be pointed out that Berlioz's data does not conform to Rosin - Rammler type kinetics and hence, strictly speaking, Eq. (2.109) is not applicable .

TABLE 9.3

Comparison of the parameter estimates reported by Mika et al. | 42 | and the reference parameters obtained in the present study.

Parameters	3300 gm. Particle Load		2640 gm. Particle Load	
	Mika et al's Estimates	Ref. Parameters	Mika et al's Estimates	Reference Parameters
S_1 [min.$^{-1}$]	0.6210	0.6128	0.6858	0.6673
$B_{2,1}$	0.406	0.473	0.389	0.473
$B_{3,1}$	0.205	0.142	0.206	0.130
$B_{4,1}$	0.098	0.106	0.109	0.102
$B_{5,1}$	0.063	0.076	0.062	0.075
$B_{6,1}$	0.044	0.055	0.048	0.056
$B_{7,1}$	0.035	0.040	0.035	0.041
$B_{8,1}$	0.023	0.029	0.027	0.031
$B_{9,1}$	0.019	0.021	0.017	0.023
$B_{10,1}$	0.013	0.015	0.016	0.017
$B_{11,1}$	0.011	0.011	0.012	0.013
$B_{12,1}$	0.010	0.008	0.009	0.009

CHAPTER 10

EMPIRICAL CORRELATIONS FOR VARIATION OF THE RATE PARAMETERS WITH PARTICULATE MASS AND BALL SIZE

In view of the fact that very little quantitative information is available at present on the breakage mechanisms operating in a ball mill , it would be useful , from the point of view of the simulation studies , if systematic procedures could be established for empirical determination of some of the more important correlations for variation of the model parameters with the operating variables . In this chapter two such methods for deriving the appropriate correlations for the effects of particulate feed charge mass and the ball size on the rate parameters have been presented . It has been assumed that the Bass model is valid for the grinding system .

10.1 Effect of Particulate Mass

As mentioned in Chapter 2 , it is not 'a priori' evident whether the influence of the particulate mass on the rate parameters is independent of particle size . In order to investigate this aspect we proceed by assuming that

it is so and the effect of the particulate mass can be incorporated in the following manner :

$$S_i(W) = H(W)\ \bar{S}_i \qquad (10.1)$$

where $S_i(W)$ is the rate parameter for i-th size interval, $\bar{S}_i$ is a characteristic constant associated with i-th size interval for a given mill and material , and $H(W)$ is a size invariant function of the particulate mass W . Our objective is to determine the function $H(W)$ from experimental data , provided , of course , Eq. (10.1) is demonstrably correct .

We rewrite the discretized batch grinding equation in the cumulative retained mode R (Eq. 2.106) as

$$\frac{dR_i(t)}{dt} = - S_i(W)\ R_i(t) + \sum_{j=1}^{i-1} \left[S_{j+1}(W)\ \bar{B}_{i,j+1} - S_j(W)\ \bar{B}_{i,j} \right] R_j(t) \quad , \ i = 2, 3, \ldots \qquad (10.2)$$

In Eq. (10.2) it has been assumed that $\bar{B}$ is independent of W |42 , 87| . Combining Eqs. (10.1) and (10.2) we obtain

$$\frac{dR_i(t)}{H(W)\ dt} = \left[- \bar{S}_i\ R_i(t) + \sum_{j=1}^{i-1} \left(\bar{S}_{j+1}\ \bar{B}_{i,j+1} - \bar{S}_j\ \bar{B}_{i,j} \right) R_j(t) \right] \qquad (10.3)$$

We next define a scaled time τ as

$$\tau = H(W)\ t \qquad (10.4)$$

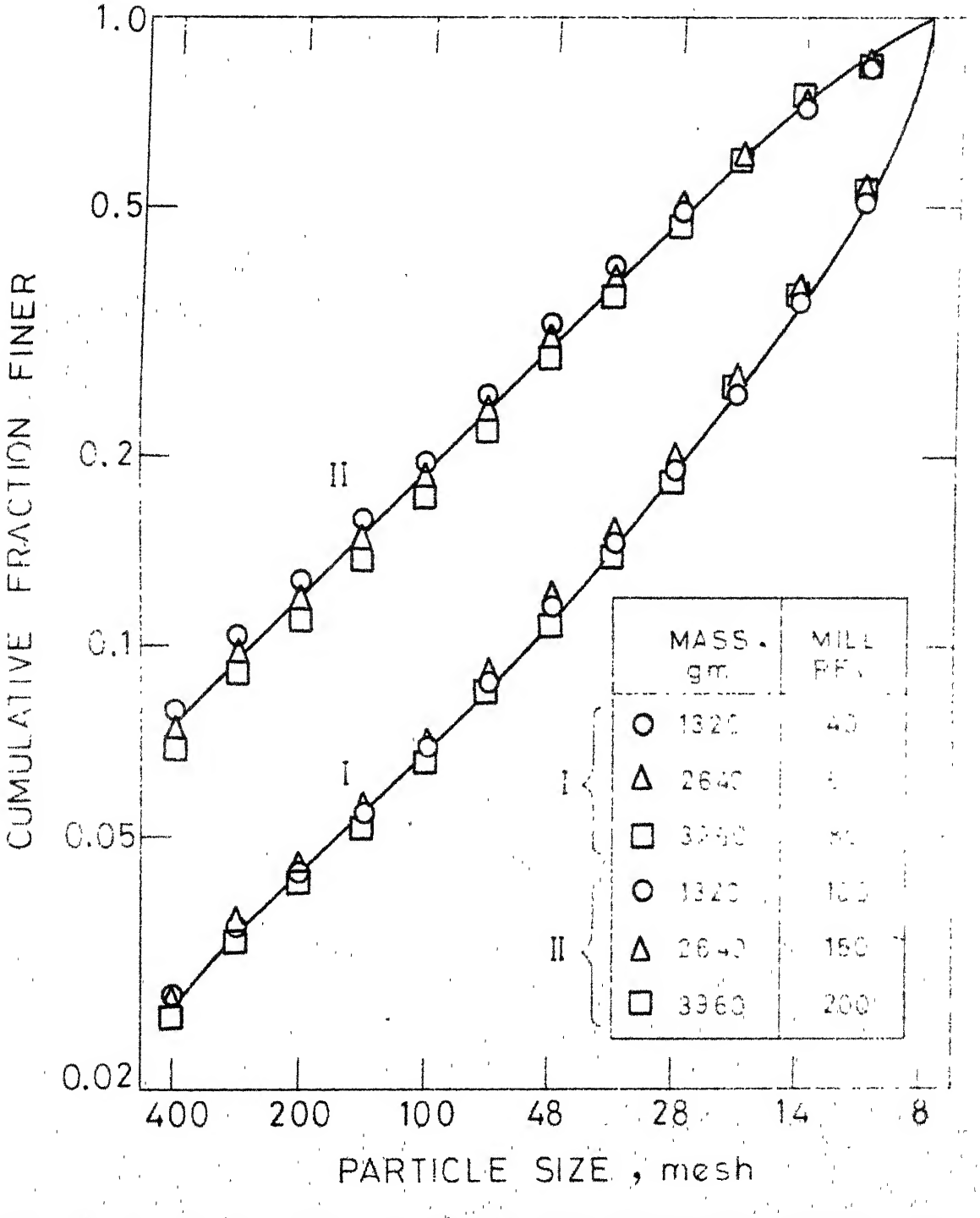
Eq. (10.3) becomes

$$\frac{dR_i(\tau)}{d} = -\bar{S}_i R_i(\tau) + \sum_{j=1}^{i-1} (\bar{S}_{j+1} \bar{B}_{i,j+1} - \bar{S}_j \bar{B}_{i,j}) R_j(\tau) \tag{10.5}$$

The solution to this equation can be formally written as

$$R_i(\tau) = X_i(\tau) \tag{10.5a}$$

It will be seen that if the time variable t is scaled by a factor H(W) , the basic form of the equation governing the batch grinding kinetics as well as its solution X_i in the domain of the scaled time variable τ is invariant of W. It then follows that the shape of tne ground particle size distributions should be identical for a given degree of fineness, independent of the particulate mass W. In other words , the trajectory of particle spectra is invariant , only the real grinding time t required to attain a specific distribution is scaled by the function H(W) . Berlioz data |108| for dolomite in Figure 10.1 shows this clearly , even though the particulate mass and the grinding time differed considerably . Rather than depend on a chance of obtaining identical distributions from experimental data , we shall now develop a systematic procedure for confirming the validity of Eq. (10.1) , which concurrently determines the



Fig·10.1 Cumulative fraction finer than the statea size for three values of the particulate mass demonstrating the invariance of the size spectra

nature of the function H(W) .

Let $\bar{r}$ be the cumulative fraction retained on the i-th size after grinding time $t_i(\bar{r})$. From Eqs. (10.4) and (10.5a)

$$H(W)\, t_i(\bar{r}) = H^o\, t_i^o(\bar{r}) \tag{10.6}$$

where the superscript denotes any arbitrary reference particulate mass . Let

$$H'(W) = H(W) \,/\, H^o$$

$$= \frac{t_i^o(\bar{r})}{t_i(\bar{r})} \tag{10.7}$$

Thus , the ratio of grinding times to reach a given cumulative fraction retained for size index i gives the numeric value of H'(W) . From Eqs. (10.1) and (10.7) we obtain

$$S_i(W) = H'(W)\, S_i^o \tag{10.8}$$

With the knowledge of the function H'(W) and the rate parameters S_i^o for the reference particulate mass , the values of the rate parameters can be obtained for any desired particulate mass W.

Berlioz data |108| for 1320, 1980, 2640, 3300, 3960, and 5280 gm. dolomite was plotted in the cumulative retained form as a function of grinding time . The reference mass chosen was 2640 gm. which resulted in near saturated

regime . For each feed charge 30-35 ratios of $t^{o}(\bar{r})/t(\bar{r})$ were computed from the graphs using all size intervals (i = 1-12) and different values of $\bar{r}$ in the range 0.3 to 0.95 . The mean and $\pm$ standard deviation bound on H'(W) are shown in Figure 10.2 . It will be seen that within the experimental and computational errors , - the latter due to graphical interpolation of data - , the values of H'(W) are reasonably constant in both the starved and over loaded regimes . Berlioz data for 660 gms. has been omitted from Figure 10.2 as it showed wide discrepancies in the values of H'(W) . . Obviously , for a highly starved mill our basic assumptions in Eqs. (10.1) and (10.2) are not valid .

In order to find the variation of H'(W), its inverse is plotted as a function of particulate mass W in Figure 10.2 . For all practical purposes this curve can be approximated by a straight line , and the equation for H'(W) becomes

$$H'W) = \frac{1}{k_1 + k_2 W} \qquad (10.9)$$

where k_1 and k_2 are constants for a given material and mill. Combining Eqs. (10.8) and (10.9) gives

$$S_i(W) = \frac{S_i^{o}}{k_1 + k_2 W} \qquad (10.10)$$

which is the required correlation for the variation of the

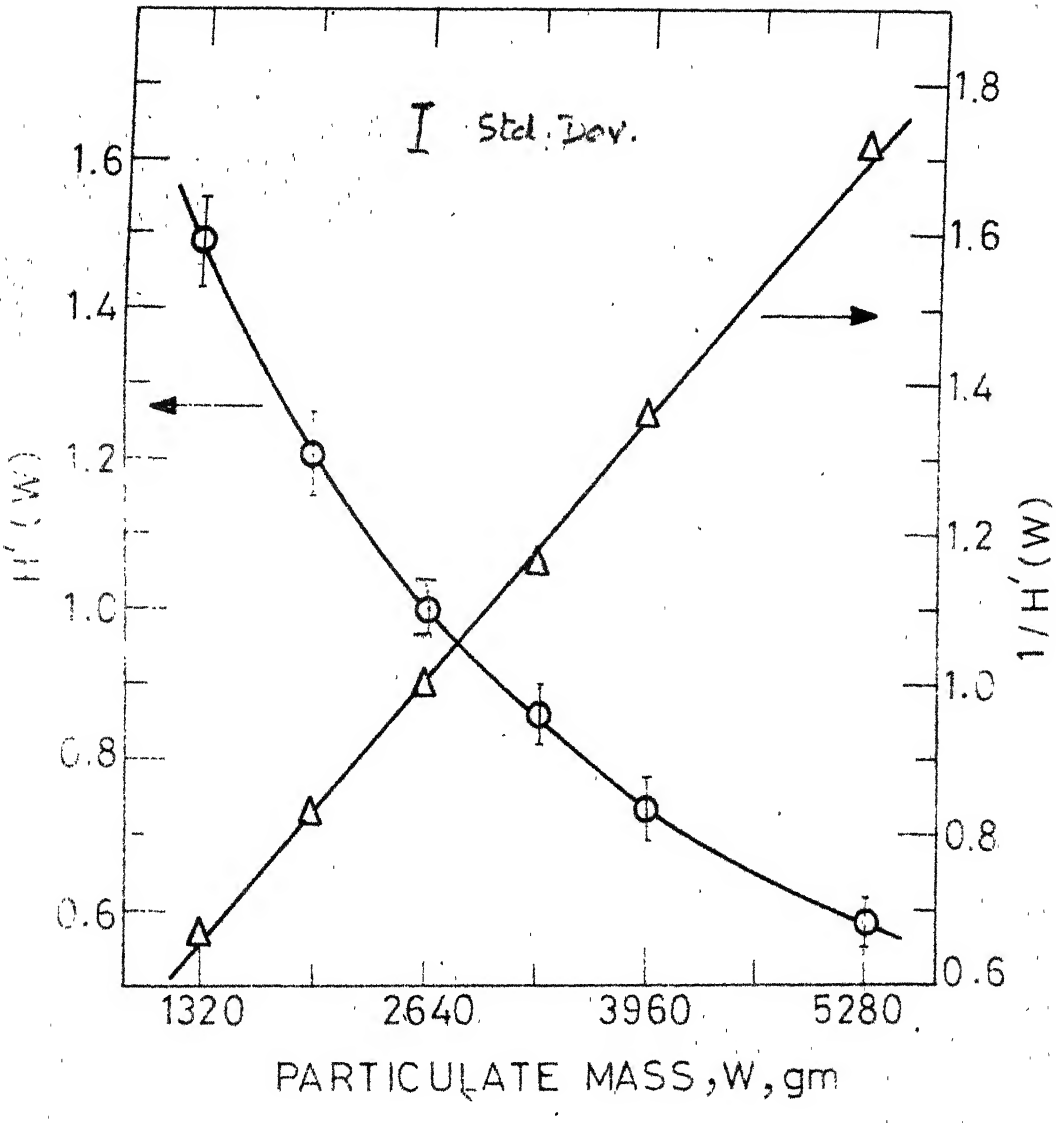


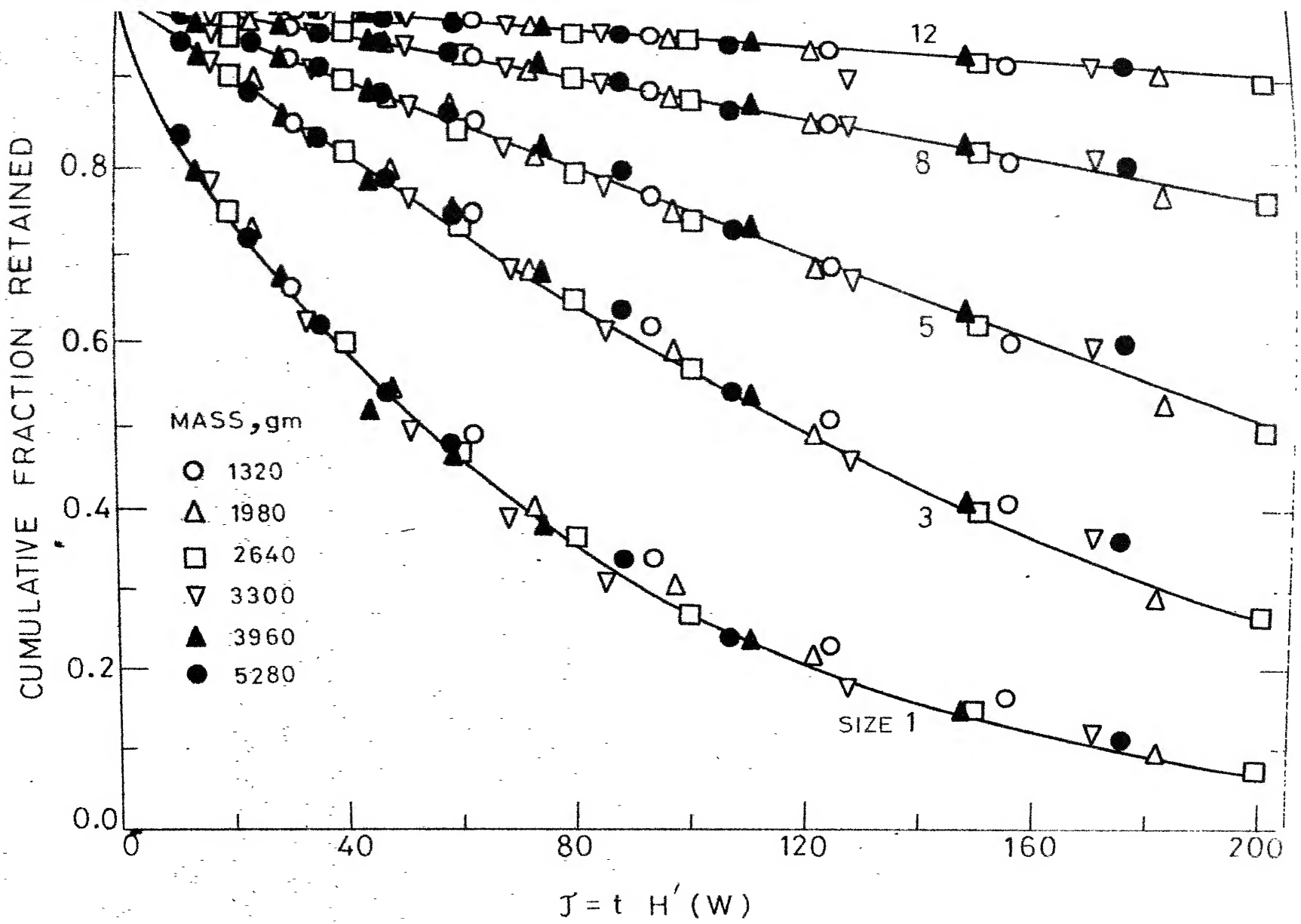
Fig. 10.2 Scaling factor $H'(W)$ and its inverse as function of the particulate mass W.

rate parameters with the mass of the particulate material . In order to confirm the validity of Eq. (10.9) , R_i for five representative sizes is plotted as a function of the scaled time $\mathcal{T}$, which was calculated from Eqs. (10.4) and (10.9) . Since the scaling factor H^o is a constant , it was arbitrarily set equal to unity for convenience . Figure 10.3 shows that the resulting curves are independent of the particulate mass, in agreement with Eq. (10.5a) .

In Eq. (10.10) the value of the constant k_1 may turn out to be of the same order as the term $k_2 W$. For Berlioz data , $k_1 = 0.3$, $k_2 = 0.265 \times 10^{-3}$, and $k_2 W$ ranges from 0.35 to 1.4 . Therefore , $S_i(W)$ is approximately inversely proportional to the particulate mass only when the latter is relatively quite large (i.e., $k_2 W >> k_1$) . It should be mentioned that the present method for determining the dependence of $S_i(W)$ on W is highly convenient , in that the more direct techniques for evaluating the data on the rate parameters as a function of hold up of the particulate mass would entail considerably greater experimental and computational efforts .

10.2 Effect of Ball Size

Careful inspection of the published data of Kelsall et al. [94] and others , suggests that there exists a unique



Fig·10.3 Cumulative fraction retained on the stated size as a function of the scaled grinding time for six different values of the particulate mass

relationship between the rate parameter $S_i(B)$ and the ball size B in the following manner

$$\frac{S_i(B)}{S_i^*} = V\left[\frac{B - B^o}{B_i^* - B^o}\right] \qquad (10.11)$$

where S_i^* is the maximum value of the rate parameter for the i-th size interval corresponding to the optimal ball size B_i^*, and B^o is a constant interpreted as the smallest ball size necessary for any meaningful grinding of the smallest particle size of interest . The function V is a unimodal frequency function . In order to obtain a general relationship it is necessary to express S^* and B^* as functions of some characteristic size of the particles lying in a size interval i. When sieve sizes differ by a constant factor δ , we are free to represent this characteristic size by the lower sieve size, upper sieve size , arithmetic or geometric mean of the lower and upper sieve sizes . In each case the ratio of the characteristic sizes for adjacent size intervals is the same . The characteristic size chosen here is the size of the openings of the lower sieve , denoted by x_i for the i-th size interval .

Let the functions describing the variations of S_i^* and B_i^* with the sieve size interval be formally represented as

$$B_i^* = U_1(x_i) \qquad (10.12)$$

and

$$S_i^* = U_2(B_i^*, x_i) \tag{10.13}$$

Substitution of Eqs. (10.12) and (10.13) into Eq.(10.11) gives

$$S_i(B) = U_2\,(U_1\,(x_i),\, x_i)\; V\left[\frac{B - B^o}{U_1(x_i) - B^o}\right] \tag{10.14}$$

which is the desired correlation for the rate parameter as a function of ball and particle size .

For demonstration data for quartz ground in a continuous ball mill were read off from Fig . 15 of Kelsall et al . [94] . Those parameters were reported as a function of five ball sizes for 6 particle size intervals , corresponding to lower size limits of 833, 590, 418, 295, 208 and 104 microns and sieve size ratio of $\sqrt{2}$. The hold-up of the particulate material in the ball mill varied with ball size as 2670, 1502, 1390, 1237 and 1216 gm. for 0.5, 0.75, 1.0, 1.25 and 1.50 inch balls , respectively . Assuming that the rate parameters were inversely proportional to the hold-up of the particulate mass , the values reported were normalized by Kelsall et al. for a standard hold-up mass of 1350 gm. In view of the conclusions arrived at in the previous section , it is likely that the corrected values of the parameters for 0.5 inch balls may be in appreciable error , if the relative magnitudes of the terms k_i and $k_2 W$ in

Eq. (10.10) were of the same order . However , for 0.75, 1.0, 1.25 and 1.5 inch balls , the maximum deviation in the hold-up mass W about the standard hold-up of 1350 gm. was about 11 per-cent only . Therefore , the normalizing factor W/1350 (W = 1502, 1390, 1237, 1216 gm.) applied by Kelsall et al. should give approximately the same results as those obtained by using the factor $\frac{k_1 + k_2 W}{k_1 + k_2 \times 1350}$, as per the correlation in Eq. (10.10)* .

The plots of the data of Kelsall et al. in Figure 10.4 show that Eqs. (10.12) and (10.13) can be written as

$$B_i^* = K_1 + K_2 x_i \qquad (10.15)$$

$$S_i^* \; B_i^* = K_3 x_i \qquad (10.16)$$

where K_1, K_2 and K_3 are constants which can be readily computed from the curve . Figure 10.5 shows the graphical form of the function V in Eq. (10.11) . On inspection it turns out that the curve can be fitted to the Cauchy distribution function | 124 | in $\log \bar{p}$, that is ,

$$V(\bar{p}) = \frac{K_5}{K_4 + (\log \bar{p})^2} \qquad (10.17)$$

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* In case of Berlioz data the two correction procedures would result in maximum 5 per-cent difference in the values of the rate parameters over the range 1200 to 1500 gm.hold-up.

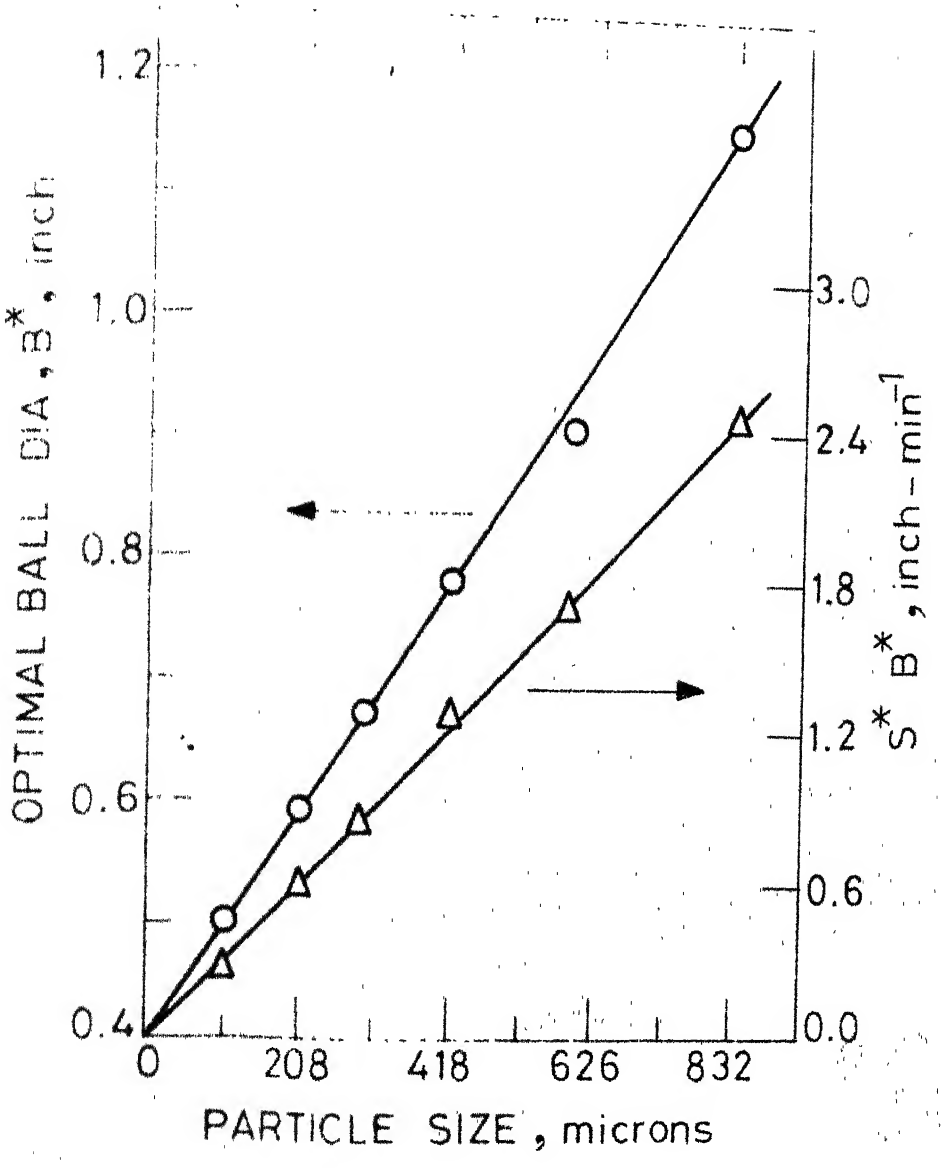

Fig. 10.4 Optimal ball diameter and the product of maximum value of rate parameter and optimal ball diameter as functions of particle size.

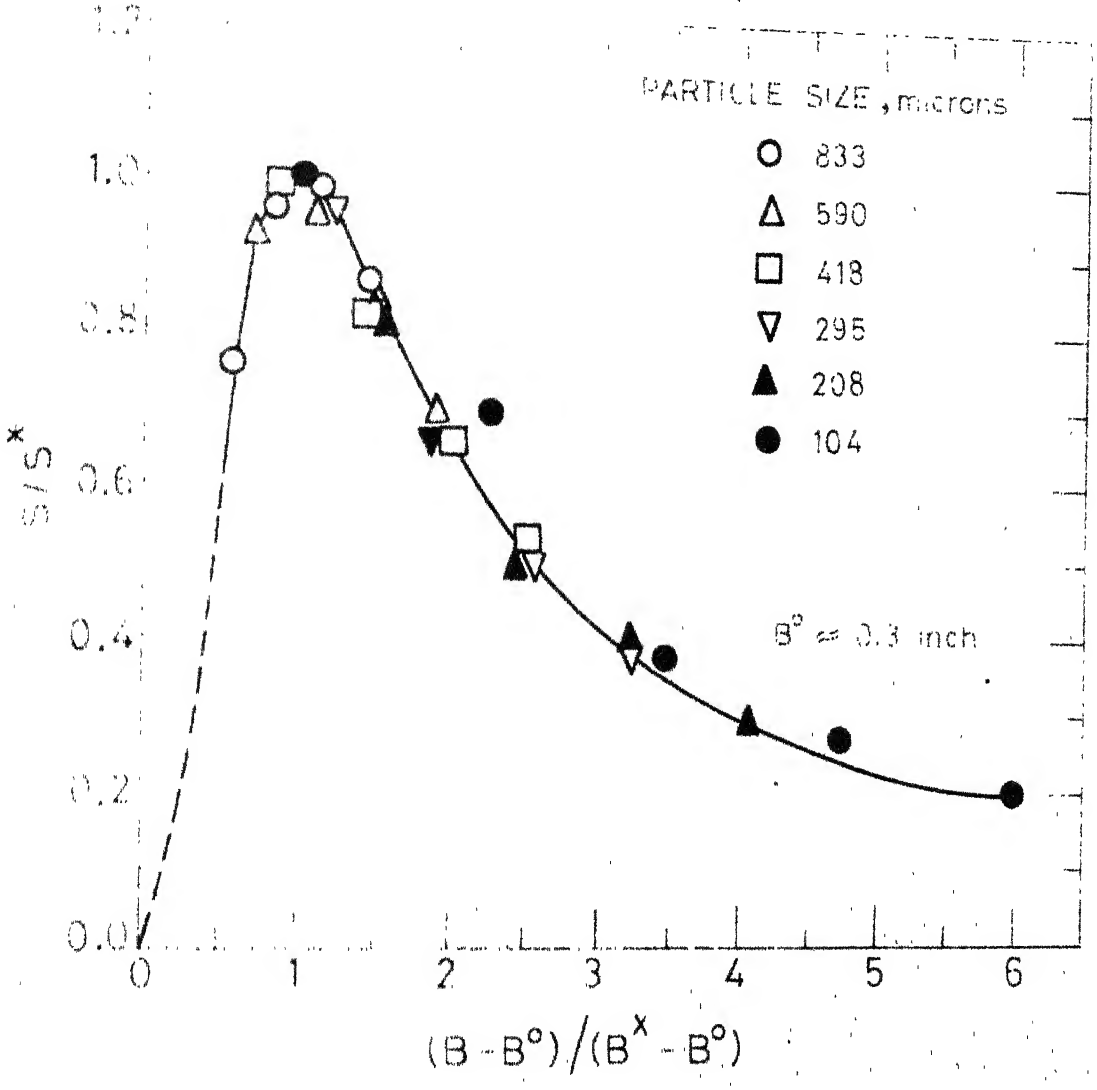

Fig 10.5 The ratio of rate parameter to its maximum value as function of variable $\bar{p}$ showing the self-preserving curve.

where K_4 and K_5 are parameters of the distribution and

$$\bar{p} = \frac{B - B^o}{B_i^* - B^o} \tag{10.18}$$

Since, for $B = B^*$, $S = S^*$ and $\bar{p} = 1$, hence, from Eqs. (10.11) and (10.17)

$$K_5 = K_4 \tag{10.19}$$

Consequently, as shown in Figure 10.6, a plot of S^*/S as a function of $(\log \bar{p})^2$ is a straight line with slope equal to $1/K_4$ and intercept 1.0. The final expression in Eq. (10.14) now becomes

$$S_i(B) = \frac{K_3 \; K_4 \; x_i}{[K_1 + K_2 \; x_i] \; [K_4 + (\log \bar{p})^2]} \tag{10.20}$$

In Figure 10.7 the family of curves has been back calculated from Eq. (10.20) and compared with the experimental data. With the exception of the data points pertaining to 0.5 inch balls, which was to be expected for reasons mentioned above, the agreement in general is quite satisfactory.

10.3 Conclusions

We have shown, at least for the grinding system investigated, that to a first approximation the rate parameters for different size intervals are influenced by the

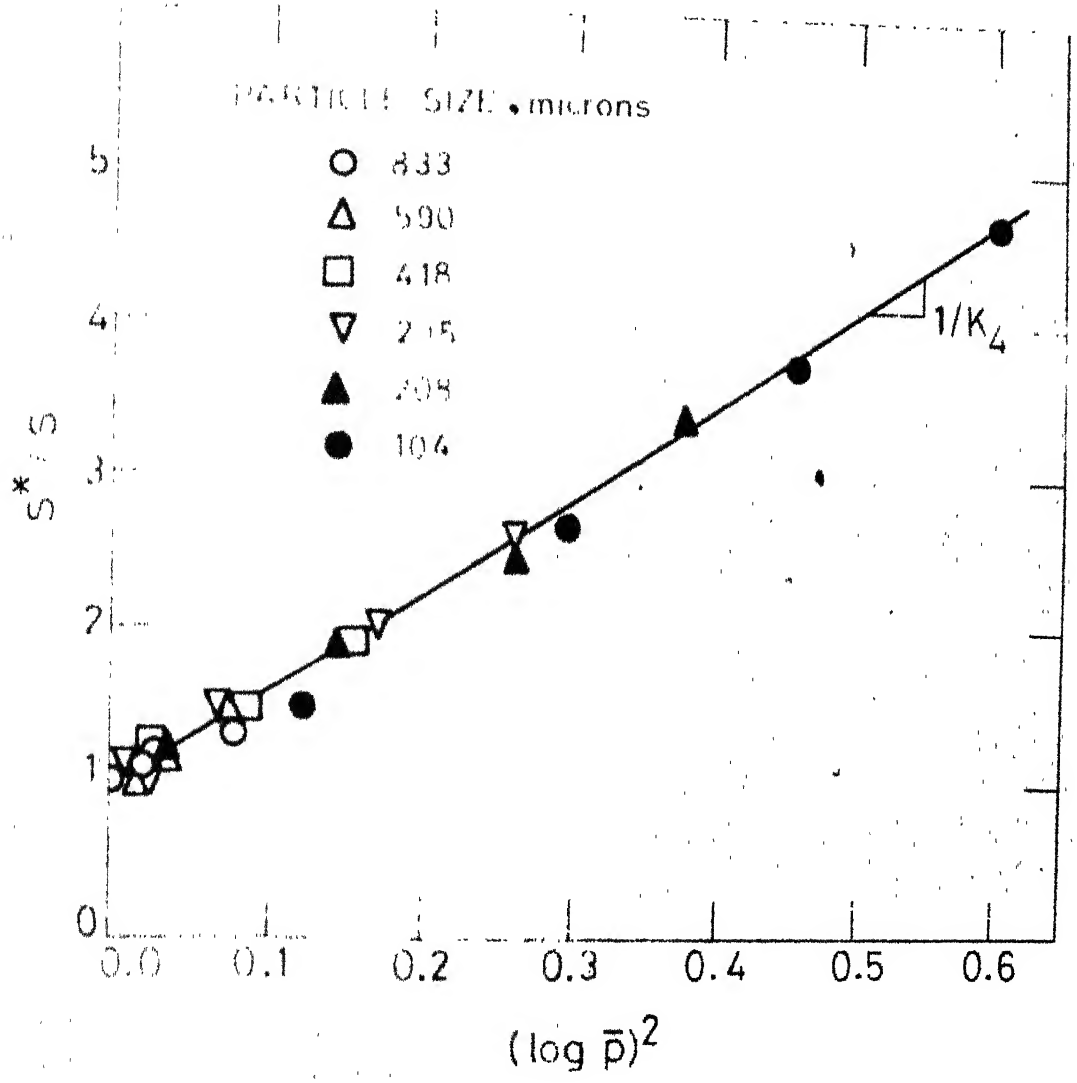

Fig.10.6 Inverse of the ratio of rate parameter to its maximum value as a function of $(\log \bar{p})^2$ for estimation of constant K_4 in the cauchy distri-bution.

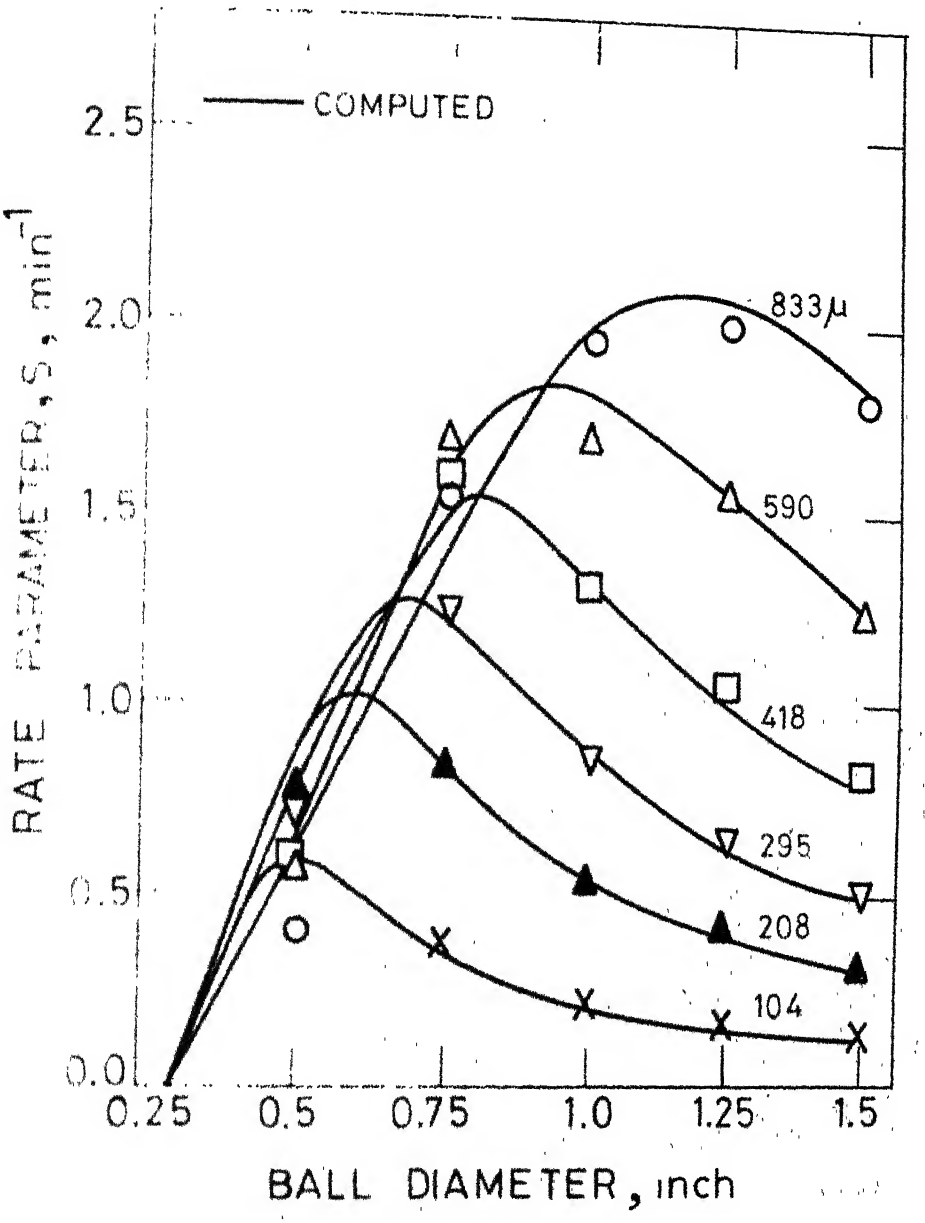

Fig.10.7 Comparison of experimental and predicted values of rate parameters for different ball sizes.

particulate mass to the same extent, in the manner of Eq. (10.1) . The parameter $S_1(W)$ is related to the particulate mass W as in Eq. (10.10) , which is found to be valid over both the unsaturated and saturated regimes , except when the mill is highly starved . The inverse proportionality between the rate parameter and particulate mass , as suggested by earlier authors , is valid only in the limit for highly saturated mills . Due to lack of appropriate data , it has not been possible to determine if the function H(W) depends on the ball size or not , although the latter was implicitly assumed for the analysis of the data of Kelsall et al.

The self preserving curve in Figure 10.5 , where S/S^* is plotted as a function of $(B-B^0)/(B^*-B^0)$, simplifies considerably our search for a reasonably accurate correlation for the effect of the ball size . The choice of the Cauchy distribution is convenient for graphical computation of its parameter K_4 , but it is by no means unique . It is possible that other distributions can be fitted to the self-preserving curve .

Finally, it is recognised that the nature of the functions H(W) , U_1 , U_2 and V are specific to the grinding system investigated and there is no 'a priori' reason why these should remain invariant for other systems . However ,

we have presented details for checking the assumptions made in Eqs. (10.1) , (10.2) and (10.11) , and given simple graphical procedures for determining the empirical correlations for the effects of particulate mass and ball size on the rate parameters from suitable experimental data .

CHAPTER - 11

SUMMARY AND CONCLUSIONS

A new solution has been obtained to the fundamental integro-differential equation of batch grinding for selection and breakage functions of the form given in Eqs. (4.1) and (4.2) , respectively. The solution given in Eq. (4.13) is the most general closed form analytical solution available at present. It describes the complete trajectory of particle spectra in time, apart from a small initial period. Unlike some commonly used self-preserving size distribution equations such as Gate-Gaudin-Schumann distribution and Rosin-Rammler-Bennett distribution |13| , Eq. (4.13) has a non-self-preserving structure . This point has been illustrated by computer simulation experiments for two different forms of the basic function $g(x)$ in Eqs. (4.1) and (4.2) .

2. Using a formal mathematical discretization procedure, exact expressions have been obtained for the parameters of the two size-discretized models in Eqs. (2.86) and (5.10) in terms of the three basic functions $S(x)$, $\bar{B}(v,x)$ and $M(x,t)$ of the

size continuous, linear and time - invariant mathematical model. Equations (5.7) , (5.8) , (5.13) , (5.15) and (5.17) clearly show that all the size discretized parameters are uniquely interrelated through these functions and therefore, cannot be assigned any arbitrary values. Moreover, because of the dependence of these parameters on the mass density function $M(x,t)$, these parameters are implicit functions of grinding time and hence are non-unique.

The product term $S(v)\ \bar{B}(v,x)$ occurs repeatedly in the analysis of discrete size models. It was felt that it deserves a name of its own. In this work it has been called as 'Selekage Function' and denoted by $L(v,x)$. Its important properties have also been delineated.

3. It has been shown that by specializing the functional forms forthe functions $M(x,t)$ and $L(v,x)$, some simplified size discretized models can be obtained. For sufficiently narrow sieve size intervals, it is a fair approximation to assume that the particle size distribution is uniformly distributed in a piece-wise manner, over the

individual discrete size interval size ranges. This gives us the Bass model, with time invariant discrete size parameters. With examples it has been shown that even when the sieve size ratio δ is as small as $\sqrt[4]{2}$, the inner breakage parameter is not negligibly small, as has been assumed by some of the investigators. If the selekage function does not exhibit any dependence on the parent particle size v i.e. $L(v,x) = L_2(x)$, the RSF model is obtained. This type of selekage function generates Rosin-Rammler type or Rosin-Rammler distributions. The rate and breakage parameters are time - invariant while the selection and distribution parameters are time dependent. For illustration, computed results on time variation of $b_{i,i}(t)$ for $L_2(x) = Ax$ have been graphically displayed for some selected discrete size intervals. The results show that at least for coarse size intervals, the approximation of time invariant inner breakage parameter is not valid. When the selekage function is separable i.e., $L(v,x) = L_1(v)\, L_2(x)$, the SSF model is obtained. In this model $B_{i,j}$ is time-invariant and rest of the parameters are time dependent. Time variation of $S_i(t)$ and $b_{i,i}(t)$ has been computed using Kapur's similarity solution for $L(v,x) = A\, v^{\alpha-\beta}\, x^{\beta}$ .It has been concluded that the extent of time variation is determined by size interval width as well as the dependence of the selekage function on the parent particle size v . If the selekage function is weakly dependent on v , wider size intervals can be employed in the analysis, with time invariant

parameters as a reasonable approximation. It has been further shown that when the selekage function $L(v,x_i)$ is a monotonically increasing function in size v , the disappearance kinetics plot for the discrete size interval i tends to concave upwards, and when it is a monotonically decreasing function in v , it tends to concave downwards with increasing grinding time .

4. It has been shown that, in general, it is not possible to realize an order kinetics concept for a discrete size interval. Only in case of Bass and RSF models, the disappearance kinetics of a discrete size interval is first-order, in the conventional sense. The phenomena of zero-order production of fines and Rosin-Rammler type grinding kinetics, are compatiable with only RSF model. The Alyavdin distribution need not necessarily arise from any inherent time-dependence of the selection function in Eq. (2.63) as assumed earlier. It has been shown , using synthetic size distribution data, that the Alyavdin equation could be a reasonably close approximate solution to theintegro-differential equation of batch grinding for a class of time-invariant selekage functions . The breakage parameters $B_{i,j}$ are difference-similar (normalizable) in the Bass model if and only if S(v) is proportional to v^{α} , $\bar{B}(v,x)$ can be represented as $\bar{B}(x/v)$ and the sieve size ratio δ is constant. The breakage parameters are difference - similar in the RSF and SSF models also, provided L(v,x) is proportional to x^{β} . The rate parameter

S_i is proportional to x_i^{α} in Bass and RSF models only, pro
$S(v)$ is proportional to v^{α} and $\bar{B}(v,x)$ is of the form $\bar{B}(x/$
The phenomena of difference - similar breakage parameters
a power law for size variation of the rate parameters are
always coupled. These results have clearly established tha
a given discrete size model possesses concurrent properties
none of which must be violated, otherwise serious inner
contradictions will develop.

5. A novel approach has been presented for identificat
of a linear and time invariant grinding system in which size
reduction occurs by chipping, impact fracture as well as
abrasion. A mill function $\theta(v,x,\bar{\bar{t}})$ has been defined which
represents the size distribution of the progeny particles
when a unit impulse of size v is ground for a fixed time
interval $\bar{\bar{t}}$. Starting from a generalized polynomial in x
and v, a suitable expression has been obtained for the mill
function which is consistent with the physical constraints
on the mill function. The constants in this expression have
been estimated using the standard least squares method of
fitting experimental data. Two sets of batch ball mill
grinding data for dolomite have been employed for illustration
of the identification scheme. In both the cases, the fit
to the experimental data has been found to be very good. The
role of the various grinding mechanisms has been discussed
in light of the nature of the estimated mill functions, and

been discussed .

6. An exact algorithm has been developed for back calculation of the rate and breakage parameters which makes use of Reid's solution to the size discretized batch grinding equation and a '0-1' variable suppression technique . The grinding data required is single size feeds ground for a fixed time interval $\bar{t}$, which can also be generated with the help of estimated mill function . Using this method it has been possible to compute the time variation of the rate and breakage parameters due to their dependence on the size distribution of the particles within the individual discrete size intervals. With a view to providing a standard reference for comparison of different grinding systems, a new set of parameters - 'reference parameters' have been defined. A comparison of the computed reference parameters has been made with the approximate estimates of rate and breakage parameters reported in the literature for the same grinding systems. The comparison shows that in principle the mill function approach and the proposed identification scheme is quite sound.

7. Systematic graphical procedures for determination of the empirical correlations for the variations in time independent rate parameters ('mass model) with particulate mass and the size of the grinding media balls have been developed and successfully demonstrated for two actual grinding systems.

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APPENDIX I

A Relationship Between Characteristic Particle Size and the First Moment of Particle Size Distribution ;

Kapur | 14 | has shown that under the restrictions given in Eqs. (2.56) and (2.57) a similarity solution to the integro-differential equation of batch grinding is

$$M(x,t) = \frac{1}{\mu_1(t)} \quad Z(\eta) \tag{I.1}$$

where

$$\eta = \frac{x}{\mu_1(t)} \tag{I.2}$$

The k-th moment of the particle size distribution at time t is given by

$$\mu_k(t) = \int_0^\infty M(x,t)\, x^k \, dx \tag{I.3}$$

Thus, combining Eqs. (I.1) , (I.2) and (I.3) gives

$$\mu_k = [\mu_1(t)]^k \left[\int_0^\infty Z(\eta)\, \eta^k \, d\eta\right] \tag{I.4}$$

or

$$\mu_k = \nu_k [\mu_1(t)]^k \tag{I.5}$$

where ν_k is a dimensionless constant . From Eqs. (2.4a) and (I.5) it follows that

$$\bar{x} = \nu_k^{1/k} \mu_1(t) \tag{I.6}$$

Hence, it is proved that the characteristic particle size $\bar{x}$ is proportional to the first moment of particle size distribution.

$$M(x,t) = \frac{C_o}{\mu_d(t)} \left[\frac{x^d}{\mu_d(t)}\right]^{\beta-1} \exp\left[-\frac{1}{\alpha h}\left(\frac{x^d}{\mu_d(t)}\right)^{\alpha}\right] [dx^{d-1}] \qquad (II.5)$$

In principle, this solution should be equivalent to the one obtained directly from Eq. (2.59) when α is replaced by αd and β is replaced by βd. Substituting for μ_d in terms of μ_1 from Eq. (II.4) , it can be confirmed that the solution in Eq. (II.5) really reduces to the self-similar form , and is identical to the one obtained directly from Eqs. (2.59), (2.60) and (2.61) .

APPENDIX III

A Flow Chart for Back Calculation of the Parameters in the Size Discretized Model 'A'

START

$i = 0$

(1) $i = i + 1$

$S_i = -\frac{1}{t} \ln (o_{i,i})$

$j = i$

(2) $j = j - 1$

if $j = 0$ GO TO (1)

$\kappa_{i,j} = M_i(t)$, $\begin{cases} M_k(0) = 1, & k = j \\ M_k(0) = 0, & k \neq j \\ B_{i,j} = 0 \end{cases}$

$C_{i,j} = M_i(t) - \kappa_{i,j}$ $\begin{cases} M_k(0) = 1, & k = j \\ M_k(0) = 0, & k \neq j \\ B_{i,j} = 1 \end{cases}$

$B_{i,j} = (\kappa_{i,j} - \kappa_{i,j})/e_{i,j}$

if $i = n$ and $j = 1$, GO TO (3)

if $j > 1$, GO BACK TO (2)

if $j = 1$, GO BACK TO (1)

(3) STOP

END

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